1. Let e_i be ith unit vector.

Consider the primal dual pair:

$$\begin{array}{llll} \min & -e_i^T x & & \max \ 0 \\ \text{s.t.} & Ax = 0 & (P_i) & \text{s.t. } A^T y \leq -e_i \quad (D_i) \\ & x \geq 0 & & \end{array}$$

If x_i is bounded in (P) then (P_i) has finite optimal value, so (D_i) is feasible, so there is a dual ray d^i with $A^T d^i \leq -e_i$.

Let $(\bar{y}, \bar{s})$ be feasible in (D). Let $s^i = -A^T d^i \geq e_i$.

Then $y = \bar{y} + \alpha d^i$, $s = \bar{s} + \alpha s^i$ is feasible in (D) for any $\alpha \geq 0$.

As $\alpha \to \infty$, we have $s_i \to \infty$.

2. Diagonalize $Q = U \Lambda U^T$, where U is orthonormal and Λ is diagonal.

Let $u^1, \ldots, u^n$ be columns of U, let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of Q.

Then $Q = \sum_{i=1}^{n} \lambda_i u^i u^{i^T}$. Note that $Q = \nabla^2 L(x, u)$

Let d satisfy the $n-k+p < n$ linear constraints:

$$u^{i^T} d = 0 \quad \forall i \text{ with } \lambda_i \geq 0$$
$$a^{j^T} d = 0 \quad \forall \text{ active constraints } j,$$

where a^{j^T} is jth row of A.

There is a nonzero d satisfying this system, since the system is underdetermined.

For such a d, we have $d^T Q d = \sum_{i=1}^{n} \lambda_i (u^{i^T} d)^2$,

so $d^T Q d = \sum_{\lambda_i < 0} \lambda_i (u^{i^T} d)^2$

Since the vectors u^i form a basis for $\mathbb{R}^n$, we have $u^{i^T} d \neq 0$ for some i, so $d^T Q d < 0$.

This d satisfies $d^T a^j = 0 \quad \forall$ active constraints j

$d^T Q d < 0$, .

So the second order necessary conditions do not hold.

3.(a) KKT points satisfy:

$$4x_2 - 1 - u_0 - s_1 + t_1 = 0 \quad (1)$$
$$4x_1 - 2 + u_0 - s_2 + t_2 = 0 \quad (2)$$
$$u_0(-x_1 + x_2) = 0 \quad (3)$$
$$s_1 x_1 = 0, \quad s_2 x_2 = 0 \quad (4)$$
$$t_1(1 - x_1) = 0, \quad t_2(1 - x_2) = 0 \quad (5)$$
$$u_0, s_1, s_2, t_1, t_2 \geq 0 \quad (6)$$
$$-x_1 + x_2 \leq 0 \quad (7)$$
$$0 \leq x_1, x_2 \leq 1 \quad (8)$$

Note that (4), (5) $\Rightarrow$ $s_1 t_1 = 0$, $s_2 t_2 = 0$ (9)

Break into cases:

Assume $x_2 = 0$: $\xRightarrow{(1)} t_1 > 0 \xRightarrow{(5)} x_1 = 1 \xRightarrow{(3),(4)} u_0 = 0,\ s_1 = 0 \xRightarrow{(1)} t_1 = 1$

Also $\xRightarrow{(2),(9),(5)}$ $s_2 = 2,\ t_2 = 0$

So get KKT point: $\boxed{x = (1, 0)}$, $u_0 = 0$, $s = (0, 2)$, $t = (1, 0)$.

Assume $x_2 > 0$: $\xRightarrow{(4)} s_2 = 0$

Assume $u_0 > 0$: $\xRightarrow{(3)} x_1 = x_2 \xRightarrow{(4)} s_1 = 0$

Assume $x_2 < 1$. Then $\xRightarrow{(5)} t_1 = t_2 = 0 \xRightarrow{(1),(2)} \boxed{x_1 = x_2 = \frac{3}{8}}$

KKT point: $u_0 = \frac{1}{2}$, $s = (0, 0)$, $t = (0, 0)$

Assume $x_2 = 1$ $\xRightarrow{(2)} u_0 + t_2 = -2$ ⨳.

So can take $u_0 = 0$, along with $x_2 > 0$ and $s_2 = 0$.

$\overset{2}{\Rightarrow} 4x_1 + t_2 = 2 \Rightarrow x_1 \le \frac{1}{2}. \overset{5}{\Rightarrow} t_1 = 0$

$\overset{1}{\Rightarrow} 4x_2 = 1 + s_1 \Rightarrow x_2 \ge \frac{1}{4}.$

Assume $x_1 < \frac{1}{2} \overset{2}{\Rightarrow} t_2 > 0 \overset{5}{\Rightarrow} x_2 = 1 \cancel{\overset{1}{\Rightarrow} s_1 > 0 \overset{4}{\Rightarrow} x_1 = 0}$

~~Get KKT point~~ $\boxed{\cancel{x = (0,1)}}$ $\cancel{u_0 = 0,\ s = (3,0),\ t = (0,2)}$
Violates 7.

Can take $x_1 = \frac{1}{2}$.
~~Assume.~~ $\cancel{x_2 > \frac{1}{4}}$.
$\overset{4}{\Rightarrow} s_1 = 0 \overset{1}{\Rightarrow} x_2 = \frac{1}{4}.$

Get KKT point $\boxed{x = \left(\frac{1}{2}, \frac{1}{4}\right)}$, $u_0 = 0$, $s = (0,0)$, $t = (0,0)$.

Second order sufficient conditions:

$$D^2 L = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}.$$

At all our 4 points, KKT multipliers of active constraints are positive.

$x = (1,0)$: ~~Let d =~~ The only d with $a_j^T d = 0$ $\forall$ active constraints is $d = 0$.

So vacuously this point satisfies 2nd order sufficient conditions.

$x = \left(\frac{3}{8}, \frac{3}{8}\right)$: Only possible d is $\pm(1,1)$

Then $d^T \nabla^2 L\, d = 8$, so this point satisfies the second order sufficient conditions.

~~$x = (0,1)$: Only valid d is $d = (0,0)$, so again~~

~~2nd order conditions are satisfied vacuously.~~

$x = \left(\frac{1}{2}, \frac{1}{4}\right)$: Any d is possible. Take $d = (1,-1)$.
Then $d^T Q d = -8$, so second order sufficient conditions violated (as are the 2nd order necessary conditions).

(b) CQ holds everywhere, since constraints linear.
Feasible region is compact.
So best KKT point is optimal soln.

$x = (1,0)$: Value is -1

$x = \left(\frac{3}{8}, \frac{3}{8}\right)$: Value is $\frac{9}{16} - \frac{9}{8} = -\frac{9}{16}$

$x = \left(\frac{1}{2}, \frac{1}{4}\right)$: Value is $\frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}$

So optimal soln is $\boxed{x = (1,0)}$, value -1.

(c) (i) $Q = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$ has eigenvalues ± 4.
So # negative eigenvalues $= 1$.

If x is not on the boundary then # active constraints $= 0$.

So x not on boundary $\overset{Q2}{\Longrightarrow}$ x is not optimal.

(ii) $\min\ 4x_1x_2 - x_1 - 2x_2 + u(-x_1 + x_2)$ $\quad (LR(u))$
s.t. $0 \le x_1, x_2 \le 1$

Suffices to find best point on boundary.
So find best points on each of four boundaries, and compare.

$x_1 = 0$: $\min\ (u-2)x_2$ s.t. $0 \le x_2 \le 1$: Optimal value $= \begin{cases} u-2 & \text{if } u \le 2, \text{ achieved by } x_2 = 1 \\ 0 & \text{if } u = 2, \text{ achieved by } x_2 \in [0,1] \\ 0 & \text{if } u > 2, \text{ achieved by } x_2 = 0 \end{cases}$

$x_2 = 0$: $\min\ (u-1)x_1$ s.t. $0 \le x_1 \le 1$: Optimal value $= \begin{cases} u-1 & \text{if } u \le 1, \text{ achieved by } x_1 = 1 \\ 0 & \text{if } u = 1, \text{ achieved by } x_1 \in [0,1] \\ 0 & \text{if } u > 1, \text{ achieved by } x_1 = 0 \end{cases}$

$x_1 = 1$: $\min\ (2+u)x_2 - 1 - u$ s.t. $0 \le x_2 \le 1$: Optimal value $= \Big\{ -1-u$ for all u, achieved by $x_2 = 0$

$x_2 = 1$: $\min\ (3-u)x_1 + u - 2$ s.t. $0 \le x_1 \le 1$: Optimal value $= \begin{cases} u-2 & \text{if } u \le 3, \text{ achieved by } x_1 = 0 \\ u-2 & \text{if } u = 3, \text{ achieved by } x_1 \in [0,1] \\ 1 & \text{if } u > 3, \text{ achieved by } x_1 = 1. \end{cases}$

Combining these calculations, we get

$$\theta(u) = \begin{cases} u-2 & \text{if } u \le \frac{1}{2} \quad \text{achieved by } x=(0,1) \\ -1-u & \text{if } u \ge \frac{1}{2} \quad \text{achieved by } x=(1,0) \end{cases}$$

(iii) Optimal dual value: $\theta(u^*) = -\frac{3}{2}$, achieved at $u=\frac{1}{2}$.

(iv) At $u=\frac{1}{2}$, have $x=(0,1)$ or $x=(1,0)$.
These give $g(x)=1$ and $g(x)=-1$.

So subdifferential is $[-1,1] \ni 0$.

4. If f is differentiable, we have

$$\nabla_x g(x,s) = \nabla f\left(\tfrac{1}{s}x\right)$$

and
$$\frac{\partial g}{\partial s} = f\left(\tfrac{1}{s}x\right) - \tfrac{1}{s}x^T \nabla f\left(\tfrac{1}{s}x\right)$$

In the general case, we want to find a subgradient of $g(x,s)$.

Thus, $g(x,s)$ is convex $\Longleftrightarrow$ $\exists$ subgradient at each point $(\bar{x},\bar{s})$ so that

$$g(x,s) \le g(\bar{x},\bar{s}) + \xi^T(x,s).$$

Let $\bar{\xi}$ be a subgradient of f at the point $\frac{1}{\bar{s}}\bar{x}$,

so $$f(x) \le f\left(\tfrac{1}{\bar{s}}\bar{x}\right) + \bar{\xi}^T\left(x - \tfrac{1}{\bar{s}}\bar{x}\right) \quad \forall x.$$

Based on the differentiable case, we conjecture:

$$g(x,s) \le g(\bar{x},\bar{s}) + \bar{\xi}^T(x-\bar{x}) + \left[f\left(\tfrac{1}{\bar{s}}\bar{x}\right) - \tfrac{1}{\bar{s}}\bar{x}^T\bar{\xi}\right](s-\bar{s})$$

If this holds, then $g(x,s)$ is convex.

Q4 p2

We have

$$g(\bar{x},\bar{s}) + \bar{\xi}^T(x-\bar{x}) + \left[f\left(\tfrac{1}{\bar{s}}\bar{x}\right) - \tfrac{1}{\bar{s}}\bar{x}^T\bar{\xi}\right](s-\bar{s})$$

$$= \bar{s} f\left(\tfrac{1}{\bar{s}}\bar{x}\right) + \bar{\xi}^T\left[x - \bar{x} - \tfrac{s}{\bar{s}}\bar{x} + \bar{x}\right]$$

$$= s\, f\left(\tfrac{1}{\bar{s}}\bar{x}\right) + s\bar{\xi}^T\left(\tfrac{x}{s} - \tfrac{\bar{x}}{\bar{s}}\right)$$

$$= s\left[f\left(\tfrac{1}{\bar{s}}\bar{x}\right) + \bar{\xi}^T\left(\tfrac{x}{s} - \tfrac{\bar{x}}{\bar{s}}\right)\right]$$

$$\geq s\, f\left(\tfrac{1}{s}x\right) \text{ by definition of } \bar{\xi}$$

$$= g(x,s).$$

Thus, g is convex.

$$h(x_1, x_2) = x_1^2 / x_2 = x_2 \left(\frac{x_1}{x_2} \right)^2$$

Now, $f(x) = x^2$ is convex, so by first part $h(x_1, x_2)$ is convex, provided $x_2 > 0$.

$h(x_1, x_2)$ is not strictly convex:

$$\nabla^2 h = \begin{bmatrix} 2/x_2 & -2x_1/x_2^2 \\ -2x_1/x_2^2 & 2x_1^2/x_2^3 \end{bmatrix}$$

$$= 2/x_2^3 \begin{bmatrix} x_2^2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{bmatrix}$$

This is not positive definite.

Vector in null space is x, so this is candidate for checking if h is not strictly convex.

We have $h(tx_1, tx_2) = t\,h(x_1, x_2)$ for any $t > 0$,

so $h(\cdot)$ is linear along radial lines from the origin.