1.

$$\nabla f = \begin{bmatrix} -x_2x_3 \\ -x_1x_3 \\ -x_1x_2 \end{bmatrix} \qquad \nabla^2 f = \begin{bmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & -x_1 \\ -x_2 & -x_1 & 0 \end{bmatrix}$$

Active constraint is $g_0(x) = x_1 + 2x_2 + 4x_3 - 48 \le 0$

$$\nabla g_0 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \qquad \nabla^2 g_0 = 0, \text{ so } \nabla^2 L = \nabla^2 f.$$

At $\bar{x} = (16, 8, 4)$, get $\nabla^2 L = \begin{bmatrix} 0 & -4 & -8 \\ -4 & 0 & -16 \\ -8 & -16 & 0 \end{bmatrix}$

Necessary conditions:

$d^T \nabla^2 L d \geq 0 \quad \forall d$ satisfying $d^T \nabla g_0 = 0$

A basis for this set of possible $d$ is $(2,-1,0)$ and $(4,0,-1)$,

so $d = (2s+4t, -s, -t)$

Then $d^T \nabla^2 L d = (2s+4t, -s, -t) \begin{bmatrix} 0 & -4 & -8 \\ -4 & 0 & -16 \\ -8 & -16 & 0 \end{bmatrix} \begin{bmatrix} 2s+4t \\ -s \\ -t \end{bmatrix}$

$$= (2s+4t)(4s+8t) + s(8s) + t(32t)$$

$$= 8(s+2t)^2 + 8s^2 + 32t^2 \geq 0 \quad \forall s, t.$$

Sufficient conditions:

$d^T \nabla^2 L d > 0 \quad \forall d \neq 0$ satisfying $d^T \nabla g_0 = 0$, since $u_0 > 0$

Now, $d^T \nabla^2 L d = 8(s+2t)^2 + 8s^2 + 32t^2 > 0$ if $(s,t) \neq (0,0)$

~~Now~~

So, condition is satisfied.

2. (a) For any $x_1 \geq 0$, set $x_2 = x_1^2$. Then $f(x) = -x_1 \to -\infty$ as $x_1 \to \infty$.

(b) $$\phi_{\bar{x},d}(\alpha) = f(\bar{x} + \alpha d) = -\bar{x}_1 - \alpha d_1 + \left(\bar{x}_2 + \alpha d_2 - (\bar{x}_1 + \alpha d_1)^2\right)^2$$

$$= -\bar{x}_1 + \alpha d_1 + \left(\bar{x}_2 - \bar{x}_1^2 + \alpha(d_2 - 2\bar{x}_1 d_1) - \alpha^2 d_1^2\right)^2$$

$$\frac{d\phi}{d\alpha} = d_1 + 2\left(\bar{x}_2 - \bar{x}_1^2 + \alpha(d_2 - 2\bar{x}_1 d_1) - \alpha^2 d_1^2\right) \times \left(d_2 - 2\bar{x}_1 d_1 - 2\alpha d_1^2\right)$$

$$= 4\alpha^3 d_1^4 + O(\alpha^2),$$ so function is eventually increasing, for $d_1 > 0$.

$$\frac{d^2\phi}{d\alpha^2} = 2(d_2 - 2\bar{x}_1 d_1 - 2\alpha d_1^2)(d_2 - 2\bar{x}_1 d_1 - 2\alpha d_1^2) - 2d_1^2 \cdot 2\left(\bar{x}_2 - \bar{x}_1^2 + \alpha(d_2 - 2\bar{x}_1 d_1) - \alpha^2 d_1^2\right)$$

$$= 2\left(d_2 - 2\bar{x}_1 d_1 - 2\alpha d_1^2\right)^2 + 4d_1^2\left(\alpha d_1 + \bar{x}_1\right)^2 - 4d_1^2(d_2\alpha + x_2)$$

$> 0$ for sufficiently large $\alpha$.

Also, $$\phi_{\bar{x},d}(\alpha) = \alpha^4 d_1^4 + O(\alpha^3)$$
$\to \infty$ as $\alpha \to \infty$ for $d_1 > 0$.

If $d_1 = 0$ then $\phi_{\bar{x},d}(\alpha) = -\bar{x}_1 + (\bar{x}_2 + \alpha d_2)^2$,
$\to \infty$ as $\alpha \to \infty$ provided $d_2 > 0$.

So $d \geq 0$, $d \neq 0 \Rightarrow \phi_{\bar{x},d}(\alpha) \to \infty$ as $\alpha \to \infty$.

3

$$L(x,u) = -x_1 + (x_2 - x_1^2)^2 - u_1 x_1 - u_2 x_2$$

$$= -(u_1+1)x_1 - u_2 x_2 + (x_2 - x_1^2)^2$$

$$\theta(u) = \inf_{x \in \mathbb{R}^2} L(x,u)$$

$$= -\infty \quad \text{for any } u \geq 0: \text{let } x_1 \to +\infty,\ x_2 = x_1^2$$