1. $f(x) \geqslant f(\bar{x}) + \xi^T(x-\bar{x})$ for any subgradient ξ, for any $x \in \mathbb{R}^n$

Thus, $\xi^T\bar{x} - f(\bar{x}) \geqslant \xi^T x - f(x)$ $\forall x$,

so $\xi^T\bar{x} - f(\bar{x}) = \sup_x \{\xi^T x - f(x)\}$,

so ~~ξ is a subgra~~ $f^*(\xi) = \xi^T\bar{x} - f^*(\bar{x})$,

or $f(\bar{x}) = \xi^T\bar{x} - f^*(\xi)$.

Conversely, if $f(\bar{x}) = x^{*T}\bar{x} - f^*(x^*)$

then $x^{*T}\bar{x} - f(\bar{x}) = f^*(x^*)$

$\geqslant x^{*T}x - f(x)$ for any $x \in \mathbb{R}^n$

so $f(x) \geqslant f(\bar{x}) + x^{*T}(x-\bar{x})$,

so x^* is a subgradient of f at $\bar{x}$.

2. Primal dual pair:

$$\begin{array}{llr} \min & -e^T d & \\ \text{s.t.} & Ad = 0 & (P0) \\ & d \geq 0 & \end{array} \qquad \begin{array}{llr} \max & 0 & \\ \text{s.t.} & A^T\pi + s = -e & (D0) \\ & s \geq 0 & \end{array}$$

K^P bounded $\Rightarrow$ optimal value of $(P0)$ is zero ($d=0$ is feasible)

$\Rightarrow$ optimal value of $(D0)$ is zero

$\Rightarrow$ $(D0)$ is feasible, with soln π^0, s^0.

$\Rightarrow$ $\pi^0 \neq 0$, since else need $s = -e$ and $s \geq 0$.

Let $\bar{y}$ be feasible in (D).

Then $\bar{y} + \alpha\pi^0$ is feasible in (D) for any $\alpha \geq 0$.

So K^D is unbounded.

3. $A(\bar{x} + \alpha d) = A\bar{x} + \alpha A d \leq b$ for any $\alpha \geq 0$,

so $\bar{x} + \alpha d$ is feasible for any $\alpha \geq 0$.

$$c^T(\bar{x} + \alpha d) + \tfrac{1}{2}(\bar{x} + \alpha d)^T Q (\bar{x} + \alpha d)$$

$$= c^T\bar{x} + \tfrac{1}{2}\bar{x}^T Q \bar{x} + \alpha\left(c^T d + \bar{x}^T Q d\right) + \tfrac{1}{2}\alpha^2 d^T Q d$$

$$\leq c^T\bar{x} + \tfrac{1}{2}\bar{x}^T Q \bar{x} + \alpha d^T(c + Q\bar{x})$$

$$\to -\infty \text{ as } \alpha \to \infty.$$

4. $\nabla f(x) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$

KKT points: $\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} + u_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ① ②

$u_1(1+x_1) = 0$, ③ $u_2(1+x_2) = 0$ ④

$-x_1 \le 1$, ⑤ $-x_2 \le 1$ ⑥ $u_1 \ge 0$, ⑦ $u_2 \ge 0$ ⑧

Cases:

$u_1 = 0 \overset{①}{\Longrightarrow} x_2 = 0 \overset{④}{\Longrightarrow} u_2 = 0 \overset{②}{\Longrightarrow} x_1 = 0$

Satisfies the conditions, so this is a KKT point, $x = (0,0)$.

$u_1 > 0 \overset{③}{\Longrightarrow} x_1 = -1 \overset{②}{\Longrightarrow} u_2 = -1$ violates ⑧.

So no KKT point with $u_1 > 0$.

Only KKT point is origin $\boxed{\bar{x} = (0,0)}$

QP unbounded: Let $\bar{x} = (-1, 0)$, $\bar{d} = (0, 1)$.

Now, $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so $\bar{d}^T Q \bar{d} = 0$.

Also, $A\bar{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \le \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $A\bar{d} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Finally, $(c + Q\bar{x})^T \bar{d} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 < 0$.

So QP is unbounded.

5. KKT: $c + Qx + A^T y = 0$ ①

$$0 \le y \perp b - Ax \ge 0 \quad ②$$

$$x^T ① \Rightarrow c^T x + x^T Q x + x^T A^T y = 0$$

$$\overset{②}{\Rightarrow} c^T x + x^T Q x + b^T y = 0$$

$$\Rightarrow 2c^T x + x^T Q x = c^T x - b^T y$$

$$\Rightarrow c^T x + \tfrac{1}{2} x^T Q x = \tfrac{1}{2}\left(c^T x - b^T y\right).$$

So QP soln can be found by finding best KKT point (since constraints linear and feasible region bounded), which is to solve

$$\begin{aligned} \min \quad & c^T x - b^T y \\ \text{s.t.} \quad & c + Qx + A^T y = 0 \\ & 0 \le y \perp b - Ax \ge 0. \end{aligned}$$

6. There are feasible points with value <0, eg $\bar{x} = (1,1,1)$.

So any point with value 0 is not optimal. Need $\bar{x}_i > 0 \ \forall i$.

The constraints are linear, so CQ holds everywhere.

The feasible region is bounded, so the global minimizer is a KKT point.

So look for all KKT points, and take the best.

KKT conditions:

$$-x_2x_3 + u_0 - u_1 = 0 \quad (1)$$
$$-x_1x_3 + 2u_0 - u_2 = 0 \quad (2)$$
$$-x_1x_2 + 4u_0 - u_3 = 0 \quad (3)$$

$u_0(x_1 + 2x_2 + 4x_3 - 48) = 0$ (4) $\qquad u_0 \geq 0$ (5)

$u_i x_i = 0, \ i = 1,2,3$ (6) $\qquad u_i \geq 0 \quad i = 1,2,3$ (7)

$x_1 + 2x_2 + 4x_3 - 48 \leq 0$ (8) $\qquad x_i \geq 0 \quad i = 1,2,3$ (9)

Cases:

(i) $u_i > 0$ for some $i = 1,2,3$. Then $x_i = 0$, so value $-x_1x_2x_3 = 0$, so not optimal.

(ii) $u_1 = u_2 = u_3 = 0$.

(a) $u_0 = 0 \ \underset{(1)}{\Rightarrow} \ x_2x_3 = 0 \ \Rightarrow \ -x_1x_2x_3 = 0$.

(b) $u_0 > 0 \quad 2(1) - (2) \Rightarrow (x_1 - 2x_2)x_3 = 0$ (10)

$4(1) - (3) \Rightarrow (x_1 - 4x_3)x_2 = 0$ (11)

$\Rightarrow \ x_1 = 2x_2$ (12), $x_1 = 4x_3$ (13) for optimal x.

$\Rightarrow \ x_2 = 2x_3$ (14)

$\underset{(4)}{\Rightarrow} \ 4x_3 + 4x_3 + 4x_3 = 48 \Rightarrow x_3 = 4$

$\Rightarrow \ \bar{x} = (16, 8, 4)$, optimal since only KKT point > 0.

Optimal value:
$-16 \times 8 \times 4$
$= -2^9$
$= -512$.