1. Let $(x^1, y^1), (x^2, y^2)$ be two points in the epigraph.
Let $\lambda \in (0,1)$ and let $\bar{x} = \lambda x^1 + (1-\lambda) x^2$,

$\bar{y} = \lambda y^1 + (1-\lambda) y^2$.

Now,

$$
\begin{aligned}
f(\bar{x}) &= c^T\bar{x} + \tfrac{1}{2}\bar{x}^T Q \bar{x} \\
&= c^T\bar{x} + \tfrac{1}{2}(\lambda x^1 + (1-\lambda)x^2)^T Q (\lambda x^1 + (1-\lambda)x^2) \\
&= c^T\bar{x} + \tfrac{1}{2}\lambda^2 x^{1T} Q x^1 + \tfrac{1}{2}(1-\lambda)^2 x^{2T} Q x^2 \\
&\qquad + \lambda(1-\lambda) x^{1T} Q x^2 \\
&= c^T\bar{x} + \tfrac{1}{2}\lambda x^{1T} Q x^1 + \tfrac{1}{2}(1-\lambda) x^{2T} Q x^2 \\
&\qquad + \tfrac{1}{2}(\lambda^2 - \lambda) x^{1T} Q x^1 \\
&\qquad + \tfrac{1}{2}\left((1-\lambda)^2 - (1-\lambda)\right) x^{2T} Q x^2 \\
&\qquad + \lambda(1-\lambda) x^{1T} Q x^2 \\
&= \lambda f(x^1) + (1-\lambda) f(x^2) \\
&\qquad + \tfrac{1}{2}(\lambda^2 - \lambda)\left(x^{1T} Q x^1 + x^{2T} Q x^2 - 2 x^{1T} Q x^2\right) \\
&= \lambda f(x^1) + (1-\lambda) f(x^2) \\
&\qquad + \tfrac{1}{2}(\lambda^2 - \lambda)(x^1 - x^2)^T Q (x^1 - x^2)
\end{aligned}
$$

If Q is psd then $f(\bar{x}) \le \lambda f(x^1) + (1-\lambda) f(x^2)$, since $\lambda^2 - \lambda < 0$.

So f is convex.

If Q is not psd then we can pick x^1, x^2 with $(x^1 - x^2)^T Q (x^1 - x^2) < 0$,

so $f(\bar{x}) > \lambda f(x^1) + (1-\lambda) f(x^2)$, so f is not convex.

2.

For integer x, we need $x_1 + x_2 + x_3 \le 2$,

since $x_1 = x_2 = x_3 = 1 \Rightarrow 4x_1 + 5x_2 + 7x_3 > 13$.

This constraint is violated by $\bar{x}$.

3. Let $x^1, x^2 \in \mathbb{R}^n$, let $\lambda \in (0,1)$, and let

$$\bar{x} = \lambda x^1 + (1-\lambda) x^2.$$

Now, $f(b_i - a_i^T x) = \max\{0, b_i - a_i^T x, a_i^T x - b_i\}$

So
$$f(b_i - a_i^T \bar{x}) = \max\{0, b_i - a_i^T \bar{x}, a_i^T \bar{x} - b_i\}$$
$$= \max\{0, b_i - \lambda a_i^T x^1 - (1-\lambda) a_i^T x^2, (1-\lambda) a_i^T x^2 + \lambda a_i^T x^1 - b_i\}$$
$$= \max\{0, \lambda(b_i - a_i^T x) + (1-\lambda)[a_i^T x^2 + b_i], \lambda(a_i^T x^1 - b_i) + (1-\lambda)(a_i^T x^2 - b_i)\}$$
$$\leq \lambda \max\{0, b_i - a_i^T x^1, a_i^T x^1 - b_i\} + (1-\lambda)\max\{0, b_i - a_i^T x^2, a_i^T x^2 - b_i\}$$
$$= \lambda f(b_i - a_i^T x^1) + (1-\lambda) f(b_i - a_i^T x^2).$$

A sum of convex functions is convex, so $g(x)$ is convex.
In particular,

$$g(\bar{x}) = \sum_{i=1}^m f(b_i - a_i^T \bar{x})$$
$$\leq \sum_{i=1}^m \left(\lambda f(b_i - a_i^T x^1) + (1-\lambda) f(b_i - a_i^T x^2)\right)$$
$$= \lambda \sum_{i=1}^m f(b_i - a_i^T x^1) + (1-\lambda) \sum_{i=1}^m f(b_i - a_i^T x^2)$$
$$= \lambda g(x^1) + (1-\lambda) g(x^2).$$

4. (a) $\frac{df}{dx} = 2x - \frac{1}{x+2}$ $\quad \frac{d^2f}{dx^2} = 2 + \frac{1}{(x+2)^2} > 0$ for $x \geq 0$.

(b) $\frac{df}{dx} = 0$ when $2x(x+2) = 1$, or $2x^2 + 4x - 1 = 0$
$\Rightarrow x^* = \frac{1}{2}(-4+\sqrt{24}) = \sqrt{6}-2$

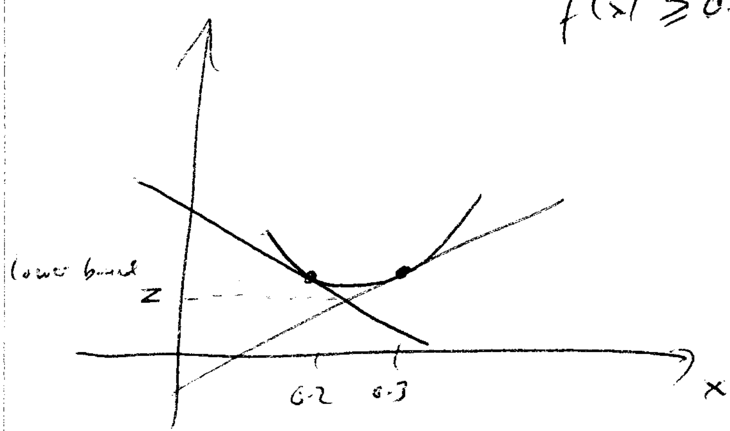
At $x = 0.2$, $\frac{df}{dx} = 0.4 - \frac{1}{2.2} = \frac{-0.12}{2.2} < 0$

At $x = 0.3$, $\frac{df}{dx} = 0.6 - \frac{1}{2.3} = \frac{0.38}{2.3} > 0$

So the minimum is between 0.2 and 0.3.

Get valid inequalities: $f(x) \geqslant 0.04 - \ln(2.2) + (x-0.2)\left(\frac{-0.12}{2.2}\right)$

$f(x) \geqslant 0.09 - \ln(2.3) + (x-0.3)\left(\frac{0.38}{2.3}\right)$



Find where lines cross to get lower bound

Solve $z = 0.04 - \ln(2.2) - \frac{0.12}{2.2}(x-0.2)$ ①

$z = 0.09 - \ln(2.3) + \frac{0.38}{2.3}(x-0.3)$ ②

② − ① $\Rightarrow \left(\frac{0.38}{2.3} + \frac{0.12}{2.2}\right)x = \frac{0.3 \times 0.38}{2.3} + \frac{0.2 \times 0.12}{2.2} - 0.05 + \ln\left(\frac{2.3}{2.2}\right)$

Q4 p2

$$\frac{0.38}{2.3}\,1 + \frac{0.12}{2.2}\,2 \Rightarrow$$

$$\left(\frac{0.38}{2.3} + \frac{0.12}{2.2}\right)z = \frac{0.38}{2.3}\left(0.04 - \ln(2.2) + 0.2\left(\frac{0.12}{2.2}\right)\right)$$

$$+ \frac{0.12}{2.2}\left(0.09 - \ln(2.3) + \frac{0.38}{2.3}(0.3)\right)$$

5. $h(x) = g(f(x))$

Let $x^1, x^2 \in \mathbb{R}^n$, $\lambda \in (0,1)$, $\bar{x} = \lambda x^1 + (1-\lambda)x^2$

Then $h(\bar{x}) = g(f(\bar{x}))$

$\leq g(\lambda f(x^1) + (1-\lambda) f(x^2))$

since f is convex and g is nondecreasing

$\leq \lambda g(f(x^1)) + (1-\lambda) g(f(x^2))$

since g is convex

$= \lambda h(x^1) + (1-\lambda) h(x^2)$.

Thus, h is convex.

$f(x) = x^2 - 4xy + 5y^2$ is convex, since $D^2 f(x) = \begin{bmatrix} 2 & -4 \\ -4 & 10 \end{bmatrix}$

which is positive definite.

$g(y) = e^y$ is convex and nondecreasing: $\frac{d^2 g}{dy^2} = e^y > 0 \;\; \forall y$.

So, by the first part of the question,

$e^{x^2 - 4xy + 5y^2}$ is a convex function on $\mathbb{R}^2$.

6. Let $x^1, x^2 \in K$.

Since K is a cone, need to show $\lambda x^1 + \mu x^2 \in K$ for any $\lambda, \mu \geq 0$.

Now, $$\sum_{i=1}^{n-1} (\lambda x_i^1 + \mu x_i^2)^2 - (\lambda x_n^1 + \mu x_n^2)^2$$
$$= \lambda^2 \left(\sum_{i=1}^{n-1} (x_i^1)^2 - (x_n^1)^2\right)$$
$$+ \mu^2 \left(\sum_{i=1}^{n-1} (x_i^2)^2 - (x_n^2)^2\right)$$
$$+ \lambda\mu \left(\sum_{i=1}^{n-1} x_i^1 x_i^2 - x_n^1 x_n^2\right)$$

Now, let $\bar{x}^1 = (x_1^1, \ldots, x_{n-1}^1) \in \mathbb{R}^{n-1}$

$\bar{x}^2 = (x_1^2, \ldots, x_{n-1}^2) \in \mathbb{R}^{n-1}$

We have $x_n^1 \geq \|\bar{x}^1\|$ and $x_n^2 \geq \|\bar{x}^2\|$,

so $x_n^1 x_n^2 \geq \bar{x}^{1T} \bar{x}^2$

Thus $\sum_{i=1}^{n-1} (\lambda x_i^1 + \mu x_i^2)^2 - (\lambda x_n^1 + \mu x_n^2)^2 \leq 0$,

so we have a convex cone.

Let $V = \{ s \in \mathbb{R}^n : \sum_{i=1}^{n-1} s_i^2 \le s_n^2, s_n \le 0 \}$.

Claim $K^\circ = V$.

Proof ~~Let~~ For any $x \in \mathbb{R}^n$, let $\bar{x} = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Define $\bar{s}$ similarly.

Show $K^\circ \subseteq V$:

Assume $s \notin V$, show $s \notin K^\circ$.

Define $x_i = s_i$, $i = 1, \dots, n-1$

$x_n = \|\bar{x}\|$.

Then $s^T x = \bar{s}^T \bar{x} + s_n \|\bar{x}\| = \|\bar{s}\|^2 + \|\bar{s}\| s_n$

$= \|\bar{s}\| (\|\bar{s}\| + s_n)$

So $s_n \le -\|\bar{s}\|$ for $s \in K^\circ$.

But since $s \notin V$, must have $s > -\|\bar{s}\|$. So $K^\circ \subseteq V$.

Show $V \subseteq K^\circ$:

Let $s \in V$

Then $s^T x = \bar{s}^T \bar{x} + s_n x_n \le 0$ since $s_n \le -\|\bar{s}\|$,

$x_n \ge \|\bar{x}\|$.

So $s \in K^\circ$.

$\blacksquare$