A

Name:

MATH2800 Introduction to Discrete Structures

First Exam, Tuesday, September 29, 2009.

You may use one sheet of handwritten notes, but no other sources. The exam consists of six questions, and lasts fifty minutes. Please work all six problems. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No calculators are allowed. Please ring your section below:

1: Monday 9am   2: Thursday 9am   3: Monday 2pm   4: Thursday 2pm

SOLUTIONS

Q1	15	
Q2	10	
Q3	15	
Q4	15	
Q5	20	
Q6	25	
Total	100	

1. (15 points) In a certain town, 5/7 of the adult men are married to 2/3 of the adult women. Assume all marriages are monogamous (no one is married to more than one other person). Also assume there are at least 60 adult men in the town. What is the least possible number of adult men in the town? of adult women in the town? (Note: make sure that you show that your answer is the smallest consistent solution.)

Let $m = \#$ men, $w = \#$ women, $r = \#$ marriages

Then $r = \frac{5}{7} m$, $r = \frac{2}{3} w$

$\Rightarrow$ $m = \frac{7}{5} r = \frac{14}{15} w$

$\Rightarrow$ m is a multiple of 14

$\Rightarrow$ $\boxed{m = 70}$

$\Rightarrow$ $r = 50$ $\Rightarrow$ $\boxed{w = 75}$

($m = 63$ doesn't work, get $r = 45$ and $w = 72.5$)

2. (10 points) Find the greatest common divisor of 217 and 95.

$217 = 2 \times 95 + 27$

$95 = 3 \times 27 + 14$

$27 = 14 + 13$

$14 = 13 + 1$

So $\gcd(217, 95) = 1$.

Alternatively, $217 = 7 \times 31$

$95 = 5 \times 19$

3. (15 points) Find all integer solutions to the linear congruence

$$8x \equiv 5 \pmod{27}.$$

Solve $8x \equiv 1 \pmod{27}$ first.

So find $\gcd(27, 8)$:

$$\left.\begin{aligned} 27 &= 3\times 8 + 3 \\ 8 &= 2\times 3 + 2 \\ 3 &= 1\times 2 + 1 \end{aligned}\right\} \Rightarrow \begin{aligned} 1 &= 3 - 2 \\ &= 3 - (8 - 2\times 3) \\ &= 3\times 3 - 8 \\ &= 3\times(27 - 3\times 8) - 8 \\ &= 3\times 27 - 10\times 8 \end{aligned}$$

Check: $3\times 27 = 81$, $10\times 8 = 80$ ✓.

So $x = -10$ solves $8x \equiv 1 \pmod{27}$ $(*)$.

So $x = -50$ solves $8x \equiv 1 \pmod{27}$.

General solution: $x = 4 + 27k$ for any integer k.

Alternatively from $(*)$:

Want to solve

$$(-10)\,8x \equiv -50 \pmod{27}$$

or $x \equiv -50 \pmod{27}$ since $(-10)8 \equiv 1 \pmod{27}$

So general solution is $x = 4 + 27k$ for any integer k.

4. (15 points) Let $N(x)$ be the statement "x is from New York City", $R(x)$ be the statement "x has red hair", and $D(x)$ be the statement "x is taking Differential Equations this semester". The domain is the set of students in this class. Express the following statement in terms of $N(x)$, $R(x)$, and $D(x)$, quantifiers, and logical connectives:

   Some student in this class has red hair or is from New York City but is not taking Differential Equations this semester.

$$\exists x\,((R(x) \vee N(x)) \wedge \neg D(x))$$

5. (20 points) Prove the following equality using induction:

$$\sum_{i=1}^{n+1} i2^i = n2^{n+2} + 2, \text{ for all integers } n \geq 0.$$

Basis Step:

$n = 0$: $\sum_{i=1}^{n+1} i2^i = \sum_{i=1}^{1} i2^i = 1 \times 2 = 2$ ✓

$n2^{n+2} + 2 = 0 \times 2^2 + 2 = 2$

Inductive Step:

Assume $\sum_{i=1}^{k+1} i2^i = k2^{k+2} + 2$.

Then $\sum_{i=1}^{k+2} i2^i = \sum_{i=1}^{k+1} i2^i + (k+2)2^{k+2}$

$= k2^{k+2} + 2 + (k+2)2^{k+2}$

by inductive hypothesis

$= (2k+2)2^{k+2} + 2$

$= 2(k+1)2^{k+2} + 2$

$= (k+1)2^{k+3} + 2$ ✓.

6. (a) (15 points) The symmetric difference of two sets A and B is defined as $A \oplus B := (A \cup B) - (A \cap B)$. Let A, B, and C be three sets. Assume $A \oplus B = A \oplus C$. Does it follow that $B = C$? Prove or give a counterexample.

   (b) (10 points) Let A, B, and C be three sets. Is it true that $A \cap (B \cup C) = (A \cap B) \cup C$? Prove or give a counterexample.

(a) Proof:

Show $B \subseteq C$: Let $x \in B$

Two cases: (i) $x \in A \Rightarrow x \notin A \oplus B \Rightarrow x \notin A \oplus C \Rightarrow x \in C$ since $x \in A$

(ii) $x \notin A \Rightarrow x \in A \oplus B \Rightarrow x \in A \oplus C \Rightarrow x \in C$ since $x \notin A$.

Thus $B \subseteq C$.

By symmetry, we also obtain $C \subseteq B$ in exactly the same way.

Thus $\boxed{B = C}$

(b) Counterexample:

$A = \emptyset \quad B = \emptyset \quad C = \{a\}$

Then $A \cap (B \cup C) = \emptyset$

$(A \cap B) \cup C = \{a\}$.