B

Name:

MATH2800 Introduction to Discrete Structures

Second Exam, Tuesday, October 27, 2009.

You may use one sheet of handwritten notes, but no other sources. The exam consists of six questions, and lasts fifty minutes. Please work all six problems. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No calculators are allowed. Please ring your section below:

1: Monday 9am   2: Thursday 9am   3: Monday 2pm   4: Thursday 2pm

SOLUTIONS

Q1	30	
Q2	20	
Q3	20	
Q4	30	
Total	100	

1. (30 points) A biased coin is tossed repeatedly until it comes up heads. The probability of a head is 0.4. Let the random variable X denote the total number of coin tosses. (Express your answers to parts (b) and (c) as fractions or as sums of a finite number of fractions or as decimals.)

   (a) (10 points) Show that $P(X = j) = 0.4(0.6)^{j-1}$, $j = 1, \cancel{2,} 3, \ldots$.

   (b) (10 points) Find $P(X \geq 3)$.

   (c) (10 points) Find $P(X \text{ is even})$.

(a) $P(j-1 \text{ tails then } 1 \text{ head}) = (0.6)^{j-1}(0.4)$

(b) $P(X \geq 3) = 1 - P(X=1) - P(X=2)$

$$= 1 - 0.4 - 0.4 \times 0.6 = 1 - 0.4 - 0.24 = 0.36$$

(c) $P(X \text{ even}) = \sum_{k=1}^{\infty} (0.4)(0.6)^{2k-1}$ odd number of tails

$$= 0.4 \times 0.6 \times \sum_{l=0}^{\infty} (0.36)^{l}$$

$$= 0.4 \times 0.6 \times \frac{1}{1-0.36}$$

$$= \frac{24}{64} = \frac{3}{8}.$$

2. (20 points) The random variable X is the number of successful outcomes in three Bernoulli trials, where the probability of success is $1/4$. The random variable Y is the number of successful outcomes in six Bernoulli trials, where the probability of success is $1/5$. The variables X and Y are independent. What is $P(X + Y \geq 1)$? (Express your answer as a sum and/or product of fractions.)

$$P(X+Y \geq 1) = 1 - P(X=0, Y=0)$$
$$= 1 - P(X=0)\,P(Y=0)$$
$$= 1 - \left(\frac{3}{4}\right)^3 \left(\frac{4}{5}\right)^6$$

3. (20 points) Let f_n denote the nth Fibonacci number, so $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, etc. Prove that

$$f_1 + f_3 + \ldots + f_{2n-1} = f_{2n}$$

when n is a positive integer.

See Version A

4. (30 points; each part is worth ten points.) Let $x_1, \ldots, x_5$ be nonnegative integers satisfying the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16.$$

(Express your answers to each part in factorials and/or powers of integers.)

(a) How many solutions are there to the equation?

(b) How many solutions are there where each $x_i \geq 1$?

(c) How many solutions are there where each x_i is no larger than 9?

(a) Divide 16 units into 5 variables.

$$\# \text{solutions} = \binom{16+5-1}{16} = \frac{20!}{16!\,4!}$$

(b) Divide remaining 11 units into 5 variables

$$\# \text{solutions} = \binom{11+5-1}{11} = \frac{15!}{11!\,4!}$$

(c) Use complements. Note at most one $x_i \geqslant 10$.

Say $x_1 \geqslant 10$. Then let $r = x_1 - 10$.

So need solution to $r + x_2 + x_3 + x_4 + x_5 = 6$

$$\# \text{solns is } \binom{6+5-1}{6} = \frac{10!}{6!\,4!}$$

Any one of variables could be $\geqslant 10$.

So # solns to eqn with some $x_i \geqslant 10$ is $\frac{5 \times 10!}{6!\,4!}$

So required # solns is $\frac{20!}{16!\,4!} - \frac{5 \times 10!}{6!\,4!}$