A

Name:

MATH2800 Introduction to Discrete Structures

Second Exam, Tuesday, October 27, 2009.

You may use one sheet of handwritten notes, but no other sources. The exam consists of six questions, and lasts fifty minutes. Please work all six problems. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No calculators are allowed. Please ring your section below:

1: Monday 9am 2: Thursday 9am 3: Monday 2pm 4: Thursday 2pm

SOLUTIONS.

Q1	30	
Q2	20	
Q3	20	
Q4	30	
Total	100	

1. (30 points) A biased coin is tossed repeatedly until it comes up heads. The probability of a head is 0.6. Let the random variable X denote the total number of coin tosses. (Express your answers to parts (b) and (c) as fractions or as sums of a finite number of fractions.) or as decimals.
   - (a) (10 points) Show that $P(X = j) = 0.6(0.4)^{j-1}$, $j = 1, 2, 3, \ldots$.
   - (b) (10 points) Find $P(X \geq 3)$.
   - (c) (10 points) Find $P(X$ is even$)$.

(a) $P(j-1 \text{ tails then } 1 \text{ head}) = (0.4)^{j-1}(0.6)$

(b) $P(X \geq 3) = 1 - P(X=1) - P(X=2)$ ~~$P(X=3)$~~

$= 1 - 0.6 - 0.24$ ~~0.6×0.16~~

$= 0.16$

Alternatively:

$$P(X \geq 3) = \sum_{j=3}^{\infty} 0.6(0.4)^{j-1} = \sum_{k=0}^{\infty} (0.6)(0.4)^{2}(0.4)^k$$

$$= 0.6 \times 0.16 \sum_{k=0}^{\infty} (0.4)^k = 0.096 \times \frac{1}{1-0.4}$$

$$= 0.6 \times 0.16 \times \frac{1}{0.6} = 0.16$$

(c) $$P(X \text{ is even}) = \sum_{k=1}^{\infty} (0.6)(0.4)^{2k-1}$$ (get odd number of Tails)

$$= \sum_{l=0}^{\infty} (0.6)(0.4)(0.16)^l$$

$$= 0.6 \times 0.4 \times \frac{1}{1-0.16} = \frac{24}{84} = \frac{6}{21}.$$

2. (20 points) The random variable X is the number of successful outcomes in three Bernoulli trials, where the probability of success is $1/3$. The random variable Y is the number of successful outcomes in ~~three~~ four Bernoulli trials, where the probability of success is $1/5$. What is $P(X+Y \geq 1)$?

$$P(X+Y \geq 1) = 1 - P(X+Y=0)$$

$$= 1 - P(X=0)\,P(Y=0)$$

$$= 1 - \left(\frac{2}{3}\right)^3 \left(\frac{4}{5}\right)^4$$

3. (20 points) Let f_n denote the nth Fibonacci number, so $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, etc. Prove that

$$f_1 + f_3 + \ldots + f_{2n-1} = f_{2n}$$

when n is a positive integer.

Use induction:

Basis step: $n=1$: $f_2 = 1$, $f_1 = 1$, so $f_2 = f_1 + \cdots + f_{2n-1} = f_1$ ✓

Inductive step:

Assume true for $n \leq k$.

Show for $n = k+1$:

$$f_{2(k+1)} = f_{2k+2} = f_{2k+1} + f_{2k}$$

$$= f_{2k+1} + f_1 + f_3 + \cdots + f_{2k-1} \quad \text{by inductive hypothesis}$$

$$= f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1},$$

as desired.

4. (30 points; each part is worth ten points.) Let $x_1, \ldots, x_5$ be nonnegative integers satisfying the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 17.$$

(Express your answers to each part in factorials and/or powers of integers.)

(a) How many solutions are there to the equation?

(b) How many solutions are there where each $x_i \geq 1$?

(c) How many solutions are there where each x_i is no larger than 9?

(a) Place 17 units into 5 variables.

$$\# \text{ solutions} = \binom{17+5-1}{17} = \frac{21!}{4!\,17!}$$

(b) Place remaining 12 units into 5 variables

$$\# \text{ solutions} = \binom{12+5-1}{12} = \frac{16!}{12!\,4!}$$

(c) Use complement. At most one integer is ≥ 10.

Say $x_1 \geq 10$, so $x_1 = 10 + r$, say.

Then $r + x_2 + x_3 + x_4 + x_5 = 7$, and # solutions to this is

$$\binom{7+5-1}{7} = \frac{11!}{7!\,4!}$$

Could be any of the ~~#~~ x_i that is ≥ 10, so # solutions with some $x_i \geq 10$ is $5 \times \frac{11!}{7!\,4!}$

So # solutions with each $x_i \leq 9$ is $\frac{21!}{4!\,17!} - \frac{5 \times 11!}{7!\,4!}$