Renewal Theorem and Long Time Asymptotics of Renewal Processes

Thursday, November 20, 2008
12:05 PM

Reference: Karlin and Taylor 5.4 and 5.5

Beyond the development of renewal equation, the main general-purpose tool for renewal processes is the renewal theorem for describing long-time asymptotics of the solutions to the renewal equation, which in turn generally describes long-time asymptotics for statistics of renewal processes.

First we state some fundamental results about long-time asymptotics that actually don't need the renewal theorem

Central limit theorem for the number of incidents that happen over a long time interval

Reference: Resnick Theorem 3.3.2 or Karlin and Taylor Theorem 5.7.1

$$N(t) \sim N\left(\frac{t}{\mu}, \frac{\sigma^2 t}{\mu^3}\right) \quad \text{for } t \to \infty$$

renewal counting process; normal; mean; variance

$$\mu = \langle T \rangle \qquad \sigma^2 = \operatorname{var} T$$

interincident time

In many renewal process models, one is interested in associated costs or rewards.

Reward function:

$$R(t) = \sum_{j=0}^{N(t)-1} R_j$$

reward collected from jth incident

represents the reward collected up through time t.

Resnick 3.4 or Karlin and Taylor Section 5.7C

Each R_j is independent of each other and identically distributed, but can depend on interarrival times $\{T_j\}$.

For example $R_j = f(T_j)$ or $R_j = f(T_{j-1})$

For example, the cost of replacing a piece of equipment depending on whether the replacement is scheduled or if the equipment breaks prematurely.

If $\mathbb{E}\,|R_j| < \infty$, then

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}\,R}{\mu}$$

One can adapt the same idea for a continuously accumulated reward

$$C(t) = \int_0^t I(u)\,du$$

$$I(t) = I_n \qquad T_n \le t < T_{n+1}$$

$$\lim \frac{C(t)}{} = \frac{\mathbb{E}(I_n T_{n+1})}{} = \text{mean reward per incident}$$

$t \to \infty \qquad t \qquad \frac{}{\mu}$

To go beyond these elementary facts about the long-time properties of a renewal process, one typically employs the more powerful tool called the Renewal Theorem.

Renewal Theorem (Karlin and Taylor Sections 5.4 and 5.5)

Let $F(t)$ be a nondecreasing function with $F(0)=0$, $\lim_{t\to\infty} F(t) = 1$

and F is not arithmetic.

1) For any bounded function $a(t)$, the integral equation

$$\star \quad \underbrace{A(t)}_{\text{unknown}} = \underbrace{a(t)}_{\text{given}} + \int_0^t \underbrace{A(t-t')}_{\text{unknown}} \, d\underbrace{F(t')}_{\text{given}}$$

has a unique solution that is bounded on finite intervals.

2) The solution to $\star$ can be expressed as follows:

$$A(t) = a(t) + \int_0^t a(t-t')\, dM(t')$$

where $M(t)$ is the renewal function, the solution to

$$M(t) = F(t) + \int_0^t M(t-t')\, dF(t')$$

– plays the role of a Green's function

3) If $a(t) \in L^1$ $\left(\int_0^\infty |a(t)|\, dt < \infty \right)$

then $\boxed{\lim_{t\to\infty} A(t) = \frac{\int_0^\infty a(t')\, dt'}{\mu}}$

where $\mu = \langle T \rangle = \int_0^\infty (1 - F(t'))\, dt'$

4) (corollary) $\lim_{t\to\infty} M(t) - M(t-s) = \frac{s}{\mu}$

-proof of corollary: Use (3) with

$$a(t) = \begin{cases} 1 & \text{for } 0 \le t \le s \\ 0 & \text{for } t > s \end{cases}$$

We have now laid out the fundamental tools for working with renewal processes. The main procedure for analyzing a given statistic for a given renewal process model is to use a renewal argument to derive a renewal equation, then use the renewal theorem to describe the solution of the renewal equation, particularly its long-time asymptotics. Many times you will find that your renewal equationdoesn't quite satisfy the conditions of the renewal theorem, and then you have to use some tricks to massage the renewal equation into a form for which the renewal theorem works.

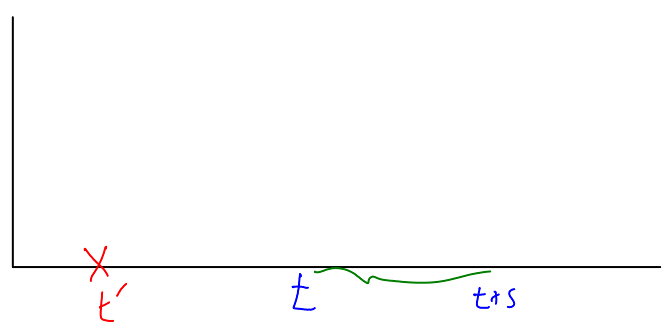
Let's see how this works for the following calculation: Given an arbitrary renewal process model with CDF $F_T(t)$ for the length of time intervals between incidents, what is the probability distribution at long times for the residual life γ_t?

$$A_s(t) = P(\gamma_t > s)$$

Renewal argument

$$P(\gamma_t > s) = \int_0^\infty P(\gamma_t > s \mid T_1 = t')\, dF_T(t')$$

$$P(\gamma_t > s \mid T_1 = t') = \begin{cases} \underbrace{P(\gamma_{t-t'} > s)}_{= A_s(t-t')} & 0 \le t' \le t \\ 0 & t < t' \le t+s \\ 1 & t' > t+s \end{cases}$$



$$A_s(t) = \int_0^t A_s(t-t')\, dF_T(t') + \int_t^{t+s} 0\, dF_T(t') + \int_{t+s}^\infty 1\, dF_T(t')$$

$$= \int_0^t A_s(t-t')\, dF_T(t') + \underset{\substack{\shortparallel \\ P(T<\infty)}}{F_T(\infty)} - F_T(t+s)$$

$$\int_0 \qquad \qquad \underset{\overset{\shortparallel}{1}}{\overset{\shortparallel}{P(T<\infty)}}$$

Renewal equation:

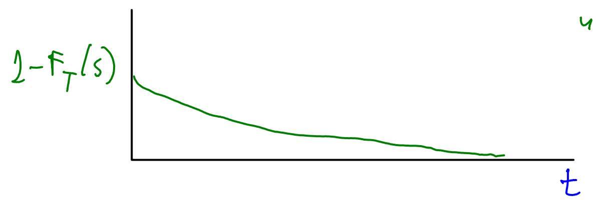
$$A_s(t) = \underbrace{1 - F_T(t+s)}_{a(t)} + \int_0^t A_s(t-t')\,dF_T(t')$$

We can analyze its solutions somewhat with the renewal theorem. Let's check the conditions. They hold because

$$0 \leq a(t) \leq 1 \quad \text{because} \quad 0 \leq F_T(t) \leq 1$$

$$\int_0^\infty |a(t)|\,dt = \int_0^\infty \left(1 - F_T(t+s)\right)dt = \int_s^\infty (1 - F_T(u))\,du \quad (u = t+s)$$

$$< \int_0^\infty (1 - F_T(u))\,du = \langle T \rangle = \mu$$



So the function $a(t)$ will be bounded and integrable provided that the time between incidents has finite mean. IF this is the case, then:

$$\lim_{t\to\infty} A_s(t) = \frac{\int_s^\infty (1 - F_T(u))\,du}{\mu}$$

$$\lim_{t\to\infty} P(\gamma_t > s) = \frac{\int_s^\infty (1 - F_T(u))\,du}{\langle T \rangle}$$

This is a general formula describing the probability distribution for the residual life given the probability distribution for the time between incidents.

One can derive similar formulas for the other key random times associated to renewal processes (Karlin and Taylor Sec. 5.6a):

$$\lim_{t\to\infty} P(\delta_t > s) = \frac{\int_s^\infty (1 - F_T(u))\,du}{\mu}$$

$$\lim_{t\to\infty} P(\beta_t > s) = \frac{1}{\mu}\int_s^\infty t'\,dF_T(t') \qquad \text{(length-biased sampling)}$$

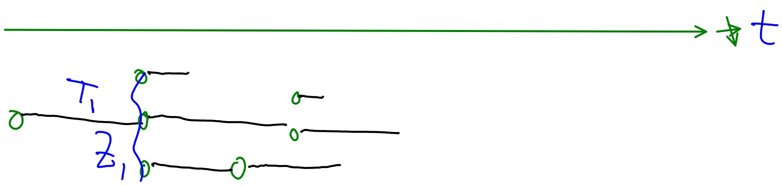
$$\langle \beta_t \rangle = \int_0^\infty t \, \frac{t \, dF_T(t)}{\mu} = \frac{\int_0^\infty t^2 \, dF_T(t)}{\mu}$$

$$\underset{\text{Cauchy-Schwarz inequality}}{\geq} \frac{\left(\int_0^\infty t \, dF_T(t)\right)^2}{\mu} = \frac{\mu^2}{\mu} = \mu$$

$$\langle \beta_t \rangle \geq \langle T \rangle$$

Continuous-time branching process (Karlin and Taylor Sec. 5.8B)

Each agent waits a random time until it produces offspring. Each offspring behaves in a statistically identical and independent manner, waiting a random time until reproducing, and then producing a random number of offspring. All branching and waiting events are independent of each other.



To specify the model, we need the CDF F_T for the time between branching events and the probability distribution for the number of offpsring Z at a branching event.

$$p_j = P(Z = j)$$

What is the long-time behavior of the size of the population? Simplest statistic is the mean:

$$\bar{X}(t) = \mathbb{E}\, X(t)$$

"population size at time t"

This continuous-time branching process isn't exactly a renewal process, but the renewal argument works with a modification.

Let's try a renewal argument trying to exploit the fact that each agent, when it is born, behaves statistically like the original parent. For simplicity we start with one agent; otherwise just multiply mean by number of initial agents.

$$\mathbb{E}\, X(t) = \int_0^\infty \mathbb{E}\left(X(t) \,\middle|\, T_1 = t' \right) dF_T(t')$$

$$\mathbb{E}\left(X(t) \mid T_1 = t'\right) = \sum_{j=0}^{\infty} \mathbb{E}\left(X(t) \mid T_1 = t', Z_1 = j\right) P(Z_1 = j)$$

$$= \sum_{j=0}^{\infty} \mathbb{E}\left(X(t) \mid T_1 = t', Z_1 = j\right) p_j$$

$$= \sum_{j=0}^{\infty} j\, \mathbb{E} X(t-t')\, p_j$$

$$= \sum_{j=0}^{\infty} j\, \bar{X}(t-t')\, p_j$$

provided $0 < t' \le t$

$$\mathbb{E}\left(X(t) \mid T_1 = t'\right) = 1 \quad \text{for } t' > t$$

Renewal equation:

$$\bar{X}(t) = \int_0^t \sum_{j=0}^{\infty} j\, p_j\, \bar{X}(t-t')\, dF_T(t') + \int_t^{\infty} 1\, dF_T(t')$$

$$\bar{X}(t) = \underbrace{1 - F_T(t)}_{a(t)} + \int_0^t \bar{X}(t-t') \underbrace{\mu_Z\, dF_T(t')}_{dF(t')}$$

$$\mu_Z = \sum_{j=0}^{\infty} j\, p_j$$

Can't quite use the renewal theorem because the function $F(t) = \mu_Z F_T(t)$

does not satisfy $\lim_{t\to\infty} F(t) = 1$

This requires a fix, which actually gives the mean growth rate of the population.

Choose $\beta > 0$ such that

$$\int_0^{\infty} e^{-\beta t}\, dF_T(t) = \frac{1}{\mu_Z}$$

This can be done provided $\mu_Z \geq 1$

because $\lim_{\beta \to 0} \int_0^\infty e^{-\beta t} dF_T(t) = 1$ #)

$\lim_{\beta \to \infty} \int_0^\infty e^{-\beta t} dF_T(t) = F_T(0) = 0$

Define $\hat{F}(t) = \mu_Z \int_0^t e^{-\beta t'} dF_T(t')$

$$\hat{\overline{X}}(t) = e^{-\beta t} \overline{X}(t)$$

Rewrite renewal eqn w/ new functions:

$$\hat{\overline{X}}(t) = e^{-\beta t}(1 - F_T(t)) + \mu_Z \int_0^t e^{-\beta(t-t')} \overline{X}(t-t')\, e^{-\beta t'} dF_T(t')$$

$$\hat{\overline{X}}(t) = e^{-\beta t}(1 - F_T(t)) + \mu_Z \int_0^t \hat{\overline{X}}(t-t')\, d\hat{F}_T(t')$$

Now $\lim_{t \to \infty} \hat{F}_T(t) = 1$

and the rest of conditions for renewal theorem apply:

$$\lim_{t \to \infty} \hat{\overline{X}}(t) = \frac{\int_0^\infty e^{-\beta t}(1 - F_T(t))\, dt}{\int_0^\infty t'\, d\hat{F}_T(t')} = c$$

$\therefore$ $\overline{X}(t) \sim c e^{\beta t}$

growth rate determined by #)