Statistical Properties of Poisson Process and Markov Chains

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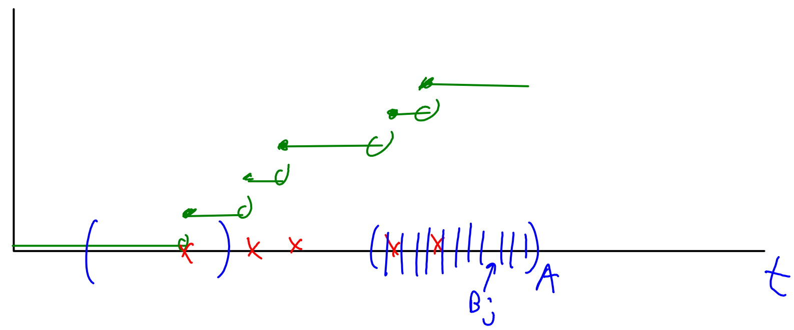
HW 3 will be posted tonight, due November 20.

Let's consider what special properties the Poisson distribution might have that explains why it arises as describing the statistics of the Poisson counting process.

Again it is connected to the Markov property.



Markov property shows that the number of events (transitions between states) must be independent in disjoint open intervals, in fact also in any disjoint Borel sets.

So let's focus on the statistics of the transition times in the Poisson counting process; this is called the Poisson point process. We have just observed that the number of Poisson points in disjoint sets must be independent.

$$A = \biguplus_{j=1}^{m} B_j \qquad \text{(disjoint union)}$$

$$N(A) = \#\ \text{points in set } A \qquad \text{(random variable)}$$

$$N(A) = \sum_{j=1}^{m} N(B_j)$$

independent r.v.

$$\langle N(A) \rangle = \mu = \lambda\, \ell(A)$$

$$\langle N(B_j) \rangle = \frac{\mu}{m} \qquad \text{if equally divided}$$

So the random variable describing the number of Poisson points in a given set must satisfy the following property:

$$(*) \qquad X^{(\mu)} \sim \sum_{j=1}^{m} X_j^{(\mu/m)}$$

independent r.v.s

for arbitrary m.

where $X^{(\mu)}$ is the probability distribution corresponding to mean μ.

That is, what we really need to describe the random number of points in any given set is to measure the amount of time in the set and then generate a random variable from the appropriate probability distribution with the parameter chosen so that the mean is μ

Random variables with the property **Q** are said to be infinitely divisible. In words, it says the the random variables corresponding to the probability distribution must be expressible as sums of arbitrary numbers of independent identically distributed copies of other random variables with corresponding scaled means. Really we think here of a one-parameter family of probability distributions parameterized by mean μ.

We can unfold what condition this imposes on the probability distributions by thinking about probability generating functions.

$$\mathcal{P}_{Y^{(\mu)}}(s) = \mathbb{E}\, s^{Y^{(\mu)}}$$

$$= \left(\mathcal{P}_{Y^{(\mu/m)}}(s)\right)^m \qquad \text{(sum of indep r.v.s)}$$

for all $m = 1, 2, 3, \ldots$

This is a rather restrictive condition but the Poisson distribution satisfies it:

$$P(Y^{(\mu)} = j) = \frac{\mu^j}{j!} e^{-\mu} \qquad j = 0, 1, 2, \ldots$$

$$\mathcal{P}_{Y^{(\mu)}} = \mathbb{E}\, s^{Y^{(\mu)}} = \sum_{j=0}^{\infty} \frac{s^j \mu^j}{j!} e^{-\mu}$$

$$= e^{-\mu} e^{s\mu} = e^{\mu(s-1)}$$

$$\left(\mathcal{P}_{Y^{(\mu/m)}}(s)\right)^m = \left(e^{(\mu/m)(s-1)}\right)^m = e^{\mu(s-1)} = \mathcal{P}_{Y^{(\mu)}}(s) \checkmark$$

The Poisson distribution is therefore infinitely divisible and therefore a natural candidate to describe the number of random points in a set when these are independent between disjoint sets and depend only on the size of the set.

Are there other probability distributions that have this property? Yes, normal distribution

$$Y \sim N(\mu, \sigma^2)$$

can always be expressed

$$Y \sim \sum_{j=1}^{m} Y_j$$

$$Y_j \sim N\left(\frac{\mu}{m}, \frac{\sigma^2}{m}\right)$$

but of course this is not a discrete nor a nonnegative probability distribution so it won't do for describing a point process.

Are there any other probability distributions which take only nonnegative integer values that are infinitely divisible?

Actually yes -- compound Poisson distributions. These are random variables which are defined to be random sums with a Poisson distributed number of terms.

$$Y^{(\mu)} = \sum_{j=1}^{N^{(\mu)}} Z_j$$

where $P(N^{(\mu)} = n) = \frac{e^{-\mu}\mu^j}{j!}$ $\quad j = 0, 1, 2, \ldots$

$\{Z_j\}$ are iid rvs, arbitrary prob. dist.

$P(Z_j = z) = p_{Z,j}$ $\quad j = 0, 1, 2, \ldots$

$$\mathcal{P}_{Y^{(\mu)}}(s) = \mathcal{P}_{N^{(\mu)}}(\mathcal{P}_Z(s))$$

$$= e^{-\mu(\mathcal{P}_Z(s)-1)}$$

$$\left(\mathcal{P}_{Y^{(\mu/m)}}(s)\right)^m = \left(e^{-\mu/m(\mathcal{P}_Z(s)-1)}\right)^m = e^{-\mu(\mathcal{P}_Z(s)-1)}$$

$$= \mathcal{P}_{Y^{(\mu)}}(s) \checkmark$$

Compound Poisson distribution corresponds to clumped distributions of points -- that is the actual times at which the incidents occur are given by a normal Poisson point process but at each of these times, the number of points is given by the probability distribution p_Z

This is good for describing situations like arrivals of groups, number of entities involved in a claim/catastrophe, births to a population.

Are there any other infinitely divisible probability distributions with discrete nonnegative values? No, these are the only ones.

If we remove the restriction that the state space of the random variable be discrete and nonnegative, then we can also get normal (Gaussian) distributions as infinitely divisible; also a class of "Levy-type" that are infinitely divisible. But that's it. (Wiener-Khintchine theorem).

This fact has some interesting implications. It's related to limit theorems.

- Central limit theorem: Sums of large numbers of independent random variables with finite variance are approximately Gaussian. (also Levy distributions arise as limits when the variances diverge.)
- Poisson limit theorem: The number of incidents is approximately Poisson distributed if the number of possible incidents is large, and each occurs rarely and independently of each other. The sum of a large number of indicator random variables, each with small probability to be true, is Poisson distributed.
  This also explains why the Poisson point process (or sometimes compound Poisson point process) appears often in modeling. Karlin and Taylor Sec. 5.9

Note that Poisson point processes can be extended to multiple dimensions -- in any desired space, one can lay down a random point pattern according to the rule that the number of points in disjoint domains is independent and the number of Poisson points in any given region is given by a Poisson (or compound Poisson) distribution with mean proportional to the area/volume of the region.

Let's now think about numerical simulations of Poisson counting process and then later the continuous-time Markov chain.

The Poisson counting process (or the associated Poisson point process) can be very simply simulated by simply simulating the times between transitions as independent exponentially distributed random variables. This is what's known as an event-based simulation scheme where, rather than laying down some sort of mesh over which time marches forward, the times at which the system is updated are chosen to be those random times at which some interesting event happens.

How do you simulate an exponentially distributed random variable? Use software like MATLAB or, by hand, the inverse transform simulation method. This is based on the idea that to simulate a random variable T with cumulative distribution function

$$F_T(t) = P(T \le t)$$

$$F_T^{-1}(U) \sim T \quad \text{where} \quad U \sim U(0,1)$$

Why does this work?

$$P(F_T^{-1}(U) \le t) = P(U \le F_T(t))$$

$$= F_T(t) = P(T \le t)$$

For exp. dist. w/mean μ $\quad F_T(t) = 1 - e^{-t/\mu}$ for $t \ge 0$

$$F_T^{-1}(u) = -\mu \ln(1-u)$$

$$T \sim -\mu \ln(1-U)$$

$U(0,1)$ uniform

By the way, this approach is also a reasonable way to simulate a Poisson distribution (with mean μ) even if one doesn't care about the full stochastic process. Simply choose a rate λ and time t such that $\mu = \lambda t$ and then simply simulate exponentially distributed random variables until their sum exceeds t; the number of exponentially distributed random variables you computed (minus 1) is the desired simulated value of the Poisson distribution.