Examples of Countable State Markov Chains

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12:12 PM

Homework 2 solutions will be posted later today.

A couple of quick examples.

Queueing model (without a preset upper bound to the queue length)

- Resnick 2.15, Lawler 2.3, Karlin & Taylor 3.5

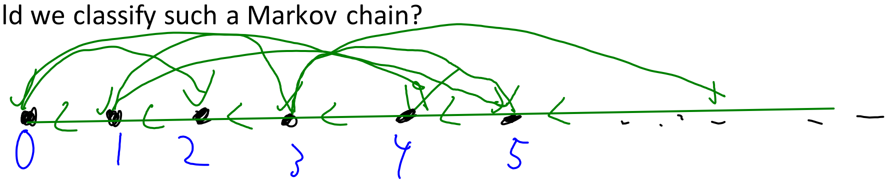
X_n is defined to be the number of requests in the queue immediately after the completion of the nth service.

Stochastic update rule: $X_{n+1} = (X_n + Z_n - 1)_+$

Z_n are iid rv's describing # customers arriving during service period.

State space $S = \mathbb{Z}_{\geq 0}$ (nonnegative integers)

How would we classify such a Markov chain?



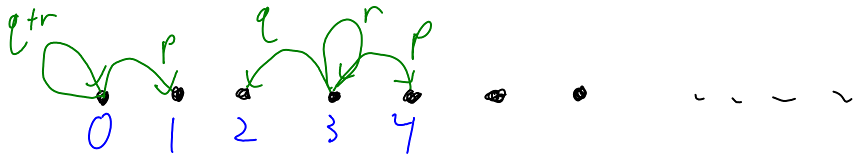
Irreducible for any model for the arrivals because from any states i and j, you can start from state i and access state 0 by decreasing by one over i epochs, and then from 0, you can access states k > j by simply following the maximum increase for sufficiently many epochs and then move to state j by decreasing by one over k-j epochs. (That is actually provided that it is possible for more than one request to arrive during a service period.)

What is the long-time property of this class? By applying the techniques from last time, one can show this Markov chain is

- transient if $\rho > 1$ (queue length grows without bound and eventually never returns to a short queue)
- positive recurrent if $0 < \rho < 1$ (queue length has a stationary distribution, meaning a finite probability to be seen at a given queue length even at large times).
- null recurrent if $\rho = 1$

$\rho = \mathbb{E} Z_n$ = expected # arrivals per departures

One-dimensional (homogenous) random walks



Here we have placed reflecting boundary conditions at the origin, but allowed the random walk to be unbounded along the positive real axis.
Had we made the origin absorbing, then the origin would be recurrent and all other states would be transient.

Working with the reflecting boundary condition as stated, the Markov chain is irreducible provided $p > 0, \; q > 0.$

Transience and recurrence properties?

- transient if $p > q$
- positive recurrent if $p < q$
- null recurrent if $p = q$

This model can be viewed as a special case of the queuing model where the probability distribution for arrivals is

$$P(Z_n = j) = \begin{cases} p & j = 2 \\ r & j = 1 \\ q & j = 0 \end{cases}$$

$$\rho = \mathbb{E} Z_n = 2p + r = 2p + 1 - p - q = p - q + 1$$

What about random walks on a lattice without boundaries?

- In one or two dimensions, random walks are null recurrent.
- In three or more dimensions, random walks are transient.

The best way to show this is actually to use explicit probability distributions for the positions of the walkers after n epochs and then use the results on transience and recurrence using the large n properties of P^n.

In physics terminology, one would say that $d=2$ is the critical dimension for random walks, separately qualitatively different behavior.

Birth-death chains:

$S = \mathbb{Z}_{\geq 0}$ $\quad\quad \Big(\; p_i \;$ for $\; j = i+1 \;\; i \geq 0$

$$P(X_{n+1}=j \mid X_n=i) = \begin{cases} r_i & \text{for } j=i,\ i>0 \\ q_i & \text{for } j=i-1,\ i>0 \\ 0 & \text{for } |j-i|>1,\ i>0 \end{cases}$$

$$\begin{cases} p_0 & j=1,\ i=0 \\ r_0 & j=0,\ i=0 \\ 0 & j\neq 1,0;\ i=0 \end{cases}$$

This has same structure as a random walk that is not homogenous in the sense that the probabilities for increasing or decreasing state value depends on the current state. Can also be thought as a basis for a model for populations or chemical species, etc. where the epochs are taken as moments of time short enough for at most one birth or death event to happen during an epoch. Random walk with reflecting boundary conditions is a special case.

How do we characterize the properties of this birth-death chain based on the transition probabilities? Think about these transition probabilities p_i, q_i, r_i as given nice functions of i.

This Markov chain will be irreducible provided:

$$p_0>0,\ p_i>0,\ q_i>0 \quad \text{for all} \quad i=1,2,\ldots$$

(If not, have to sort through special cases).

Working with the generic situation where the Markov chain is irreducible, how do we classify the Markov chain? Follow recipe from end of last lecture.

First step is to look for an invariant measure.

$$\vec{v}\cdot P = \vec{v}$$

$$v_j \geq 0 \quad \text{for } j=0,1,2,\ldots$$

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \cdots \\ \hline 0 & r_0 & p_0 & 0 & 0 & \cdots & \\ 1 & q_1 & r_1 & p_1 & 0 & \cdots & \\ 2 & 0 & q_2 & r_2 & p_2 & 0 & \cdots \\ 3 & & & & & & \\ \vdots & & & & Q & & \end{array}$$

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\
\begin{pmatrix}
r_0 & p_0 & 0 & 0 & \cdots & \\
q_1 & r_1 & p_1 & 0 & \cdots & \\
0 & q_2 & r_2 & p_2 & 0 & \cdots \\
\vdots & & & & Q & \\
\end{pmatrix}
\end{array}
$$

$$v_0 r_0 + v_1 q_1 = v_0$$

$$v_0 p_0 + v_1 r_1 + v_2 q_2 = v_1$$

$$\vdots$$

$$v_{j-1} p_{j-1} + v_j r_j + v_{j+1} q_{j+1} = v_j \quad \text{for } j \geq 1$$

How to solve these equations? Could proceed recursively but here we can use a trick -- try a detailed balance solution (which we discussed for stationary distribution but can also use it for invariant measure).

$$v_i P_{ij} = v_j P_{ji} \quad \text{for all } i,j \in S$$

If this were true, then we would have

$$v_j p_j = v_{j+1} q_{j+1} \quad \text{for } j \geq 0$$

Does this work?

$$v_{j+1} = \left(\frac{p_j}{q_{j+1}}\right) v_j$$

$$v_j = \left(\prod_{i=0}^{j-1} \frac{p_i}{q_{i+1}}\right) v_0 \quad \text{for } j \geq 1$$

This is a self-consistent solution to the detailed balance equations and therefore an invariant measure for any $v_0 > 0$.

Can we convert it to a stationary distribution by normalizing it?
Yes, if and only if

$$\sum_{j=0}^{\infty} r_j < \infty$$

$$\Leftrightarrow \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j-1} \frac{p_k}{q_{k+1}} \right) < \infty$$

If this condition is true, then we can construct a stationary distribution and the Markov chain is positive recurrent.

If this condition is not true, then might there be some other invariant measure that could be normalized to a stationary distribution? No because an irreducible Markov chain has a unique invariant measure, up to multiplication by a constant. Since we've found one such invariant measure, there are no other invariant measures, and so there are no normalizable invariant measures and the Markov chain cannot be positive recurrent.

Now we proceed to the second stage of the classification procedure to see whether the Mrakov chain is null recurrent or transient.

Let's take 0 as a reference state and solve $Qx=x$.

This is a tridiagonal system which can be solved by recursion (as we did in a previous problem).

$$x_j = x_1 \left(\sum_{k=1}^{j} \gamma_k \right)$$

$$\gamma_k = \prod_{1 \le k' \le k-1} \frac{q_{k'}}{p_{k'}}$$

Note $\gamma_k = p_0 / (r_{k-1} p_{k-1})$

The equation $Qx=x$ has a bounded nonnegative solution provided that

$$\sum_{k=1}^{\infty} \gamma_k < \infty$$

Summarizing, the birth-death chain is

- positive recurrent whenever $\sum_{j=1}^{\infty} r_j < \infty$
- transient whenever $\sum^{\infty} \frac{1}{\ } < \infty$

$$\sum_{j=1} r_j P_j$$

- null recurrent when neither of the above conditions hold.

Here $$r_j = \prod_{k=0}^{j-1} \frac{p_k}{q_{k+1}}$$