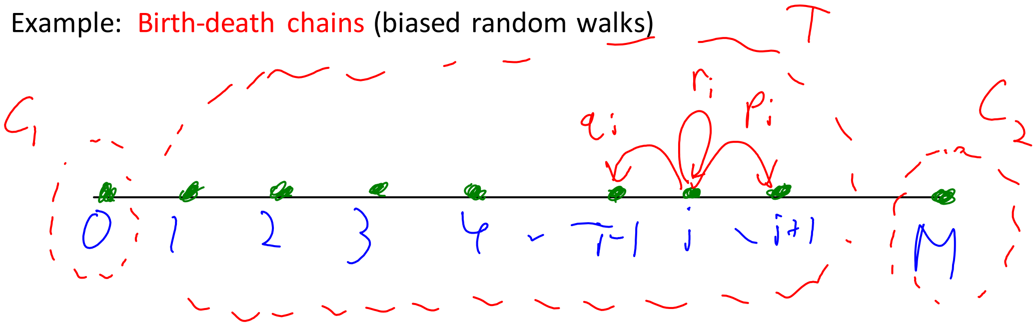
Absorption Probability Example

Monday, October 06, 2008
12:13 PM

Homework 2 will be due Tuesday, October 14 at 12 PM.

Example: Birth-death chains (biased random walks)



Generally speaking, we have:

$$p_i + q_i + r_i = 1 \qquad i = 0, 1, \ldots, M$$

$$q_0 = 0$$

$$p_M = 0$$

Special case: Absorbing boundary conditions

$$p_0 = 0 \quad r_0 = 1$$

$$q_M = 0, \quad r_0 = 1$$

Birth-death chains have natural interpretations as random walks but also apply much more generally to:

- population model
- chemical reaction models
- financial price fluctuations (if no price shocks!)

Can formulate this Markov chain model easily as a stochastic update rule, but we'll use the probability transition matrix with the purpose of setting up our absorption probability calculations.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ \\ M \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & & & M \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & \cdots & & & 0 \\ q_1 & r_1 & p_1 & 0 & \cdots & & 0 \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & & \\ & 0 & & & \ddots & & 0 \\ & & & 0 & q_{M-1} & r_{M-1} & p_{M-1} \\ & & & & 0 & 0 & 1 \end{array}\right] \end{array}$$

Suppose starting from a certain state $X_0 = i$, what is the long-time behavior of the Markov chain? Since the Markov chain is reducible with one-way accessibility between some classes, we can deduce that the Markov chain will end up in one of the recurrent classes, namely the boundary states 0 or M. Natural question: In which state does the Markov chain end up (is it absorbed)? Also, how long until the Markov chain is absorbed?

We'll show the first calculation explicitly; the second one will follow from our setup and formulas from previous lecture but we won't carry out the details.

To prepare for either of these calculations, we can write the probabilty transition matrix in canonical form (this is optional):

$$P = T \qquad \begin{array}{cc|c} & & \begin{array}{cccccccc} 0 & M & 1 & 2 & 3 & 4 & \cdots & M-1 \end{array} \\ C_1 & 0 & \\ C_2 & M & \\ & 1 & \\ & 2 & \\ & 3 & \\ & 4 & \\ & \vdots & \\ & M-1 & \end{array} \left[\begin{array}{cc|cccccc} 1 & 0 & 0 & \cdots & & & & 0 \\ 0 & 1 & 0 & \cdots & & & & 0 \\ \hline q_1 & 0 & r_1 & p_1 & 0 & \cdots & & 0 \\ 0 & 0 & q_2 & r_2 & p_2 & \cdots & & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \\ \vdots & \vdots & & 0 & \ddots & \ddots & \ddots & \\ 0 & 0 & & & & q_{M-2} & r_{M-2} & p_{M-2} \\ 0 & p_{M-1} & 0 & \cdots & & & q_{M-1} & r_{M-1} \end{array}\right]$$

$$\underbrace{}_{S} \qquad \underbrace{}_{Q}$$

We now can use the formula for absorption probabilities

$$U_{ij} = P(X_T = j \mid X_0 = i)$$

$$T = \min\{n > 0 : X_n \in \{0, M\}\}$$

$$j \in \{0, M\}$$

$$i \in \{0, 1, 2, \ldots, M\}$$

$$U = (I - Q)^{-1} S$$

This is not a convenient form for analytical computation. Since the Q matrix is relatively simple (tridiagonal), we might try for an analytical expression by simply solving the linear system for the U_{ij}. Of course, if one is interested simply in a numerical answer for specific values of the birth-death parameters, then one can use this matrix form and numerical linear algebra package (more efficient actually to have the numerical linear package solve for U through a linear system $(I-Q)U=S$

rather than inverting the matrix (MATLAB does the efficient approach automatically unless you program it not to be smart).

Analytically, we proceed from:

$$(I-Q)U=S$$

$$U_{ij} - \sum_{k=1}^{M-1} Q_{ik} U_{kj} = S_{ij}$$

$$i \in \{1,2,\ldots,M-1\}$$
$$j \in \{0, M\}$$

And since we're writing out the equations as a linear system rather than in condensed matrix form, we can simply replace entries of Q, S by correspondoing entries of P.

$$U_{ij} - \sum_{k=1}^{M-1} P_{ik} U_{kj} = P_{ij} \qquad i \in \{1,2,\ldots,M-1\}$$
$$j \in \{0, M\}$$

Because of our analytical solution procedure, we've actually undone some of the steps in the derivation of the general formula.

Now let's write down concretely what the system of equations are for our particular system with absorbing boundary conditions:

For $2 \leq i \leq M-2$,

$$U_{ij} - \underbrace{q_i U_{i-1,j} - r_i U_{i,j} - p_i U_{i+1,j}}_{\sum_{k\in T} Q_{ik} U_{kj}} = 0 = P_{ij}$$

For $i=1$:

$$U_{1j} - r_1 U_{1j} - p_1 U_{2,j} = P_{1j} = \begin{cases} q_1 & \text{for } j=0 \\ 0 & \text{for } j=M \end{cases}$$

Notice that if we just arbitrarily define:

$U_{00}=1, \quad U_{0M}=0$ even though the first index is not a transient state, we can write the i=1 equation in the same form as for the interior states

$2 \leq i \leq M-2$

Similarly, by defining $U_{M0} = 0, \; U_{MM} = 1$

we can show that the equation for i=M-1 is also of the same general form.

Therefore, the linear system(s) we need to solve for the absorption probabilties are:

$$U_{ij} = q_i U_{i-1,j} + r_i U_{ij} + p_i U_{i+1,j} \quad \text{for } i = 1, 2, \ldots, M-1, \; j \in \{0, M\}$$

Boundary conditions:

$$U_{00} = 1, \; U_{M0} = 0, \quad U_{0M} = 0, \; U_{MM} = 1.$$

Notice that this linear system is decoupled for different values of j. Therefore, we are really solving two separate linear systems, one for j=0 and one for j=M.

Solving these linear systems (technique is sort of special to these type of equations, but they arise a few places in stochastic processes…)

$$r_i = 1 - p_i - q_i$$

Rewrite:

$$q_i (U_{i-1,j} - U_{ij}) + p_i (U_{i+1,j} - U_{ij}) = 0$$

This sort of looks like a perfect "discrete derivative" that we can integrate.

$$V_{ij} = U_{ij} - U_{i-1,j} \quad \text{for} \quad 1 \leq i \leq M$$

$$-q_i V_{ij} + p_i V_{i+1,j} = 0$$

$$V_{i+1,j} = V_{ij} \underbrace{\frac{q_i}{p_i}}_{} \equiv \gamma_i$$

$$V_{i+1,j} = V_{ij}\frac{q_i}{p_i} \equiv \gamma_i$$

$$\Rightarrow V_{ij} = \boxed{\left(\prod_{1\le k\le i-1}\frac{q_k}{p_k}\right)} V_{1j}$$

To solve for V_{ij}, use boundary conditions

$$\sum_{i=1}^{N} V_{ij} = U_{Nj} - U_{0j} \quad \text{(telescoping sum)}$$

$$= \delta_{Nj} - \delta_{0j}$$

$$\left(\sum_{k=1}^{N}\gamma_k\right) V_{1j} = \delta_{Nj} - \delta_{0j}$$

$$V_{1j} = \frac{\delta_{Nj} - \delta_{0j}}{\sum_{k=1}^{N}\gamma_k}$$

$$V_{ij} = \gamma_i V_{1j} = \gamma_i(\delta_{Nj} - \delta_{0j})$$

$$U_{ij} = \sum_{k=1}^{i} V_{kj} + \underset{\delta_{0j}}{\underset{\parallel}{U_{0j}}}$$

Plug in j=0 then j=M:

$$U_{i0} = P(X_\tau = 0 \mid X_0 = i)$$
$$= \frac{\sum_{k'=i+1}^{M}\gamma_{k'}}{\sum_{k=1}^{M}\gamma_k}$$

$$U_{iM} = P(X_\tau = M \mid X_0 = i) = 1 - U_{i0}$$

$$U_{iM} = P(X_\tau = M \mid X_0 = i) = 1 - U_{i0}$$

$$= \frac{\sum_{k'=1}^{i} \gamma_{k'}}{\sum_{k=1}^{M} \gamma_k}$$

where $\gamma_i = \prod_{1 \leq k \leq i-1} \frac{q_k}{p_k}$

Expected time until absorption can be computed similarly.