

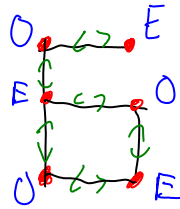
# Application of Stationary/Limit Distributions

Thursday, September 18, 2008  
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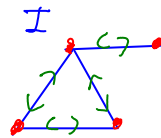
Homework 1 due Monday, September 22 at 12 PM.

How does one compute periodicity for a Markov chain?

To see why periodic Markov chains can arise, consider a random walk on the following graph, with the stipulation that the walker must move to a neighboring location at each epoch.



What is the period of this Markov chain (since it's irreducible, all states have the same period.) The random walker alternates between visiting E and O subsets, so to return to its initial location, it must take an even number of steps, and it is possible to return in 2 steps, so the greatest common denominator of all possible return times is 2. Therefore this Markov chain has period 2.



It's possible to return to the initial state  $i$  in 2 or 3 epochs, and since these are relatively prime numbers, the greatest common denominator is 1 and the Markov chain has period 1, so is aperiodic. Another simpler way to prove aperiodicity is to find some state in the class of interest which can return to itself in 1 epoch.

How does one compute the stationary distribution  $\vec{\pi}$ ?

$$\begin{aligned} \vec{\pi} \cdot P &= \vec{\pi} \\ \pi_j &\geq 0 \\ \sum_{j \in S} \pi_j &= 1 \end{aligned}$$

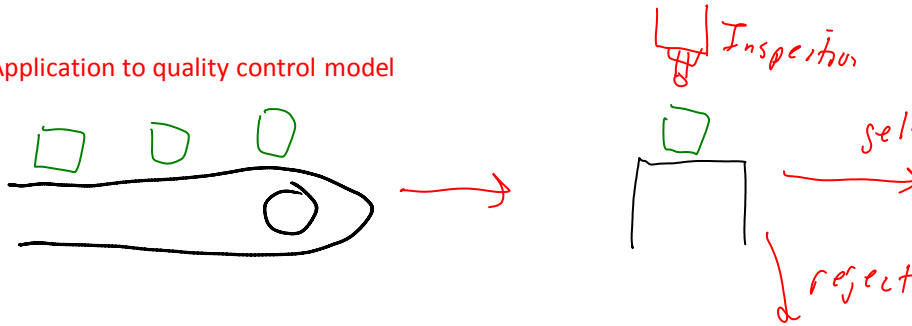
Possibilities:

- Directly compute the left eigenvector of  $P$  with eigenvalue 1 (which is the same as the right eigenvector of  $P^T$  with eigenvalue 1) and then rescale the results so the sum of the entries add up to 1. This is guaranteed to work if the Markov chain is irreducible.

$$\vec{\pi} = \vec{1} \cdot (I - P + (\cdot) N E)^{-1}$$

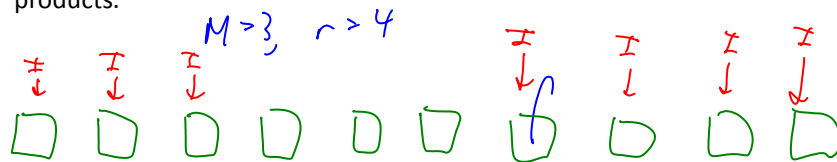


Application to quality control model



Since inspection is a costly/time-consuming process, one typically does not inspect every single product. Consider the following "sampling" protocol:

- Start by inspecting every product until the inspector sees  $M$  consecutive good products.
- After the inspector sees  $M$  consecutive good products, it then only inspects  $1$  out of every  $r$  products in regularly spaced intervals.
- Once the inspector sees a defective product, it reverts to inspecting every single product until it again sees  $M$  consecutive good products.



Uninspected products get shipped along with the inspected good products.

Practical questions:

- What fraction of shipped products are defective (in the long run), and what costs are associated with that?
- What fraction of products are inspected (in the long run), and what's the associated cost?

In addressing these questions, one of course needs to know something about the statistics of the defects in the products. For simplicity, let's suppose that each product has a probability  $p$  to be defective, independent of the defect states of other products. This might be an adequate model for nonsystemic defects, but would be a poor representation of systemic defects like a machine problem or a large quantity of corrupted material. One can extend the present techniques by enriching the Markov chain to take into account clumping of defects.

Let's setup a Markov chain model for this inspection protocol.

Look at this from the inspector's point of view:

$X_n$  = number of consecutive good products seen by the inspector right after the  $n$ th inspection

Each epoch corresponds to an inspection

$S = \{0, 1, \dots, M\}$  (don't need to keep track if more than

$M$  good consecutive products seen since doesn't affect strategy)

$$X_{n+1} = \begin{cases} X_n + 1 & \text{w/prob } 1-p \\ 0 & \text{w/prob } p \end{cases} \text{ if } 0 \leq X_n \leq M-1$$

$$\begin{cases} M & \text{w/prob } 1-p \\ 0 & \text{w/prob } p \end{cases} \text{ if } X_n = M$$

$$X_{n+1} = \begin{cases} \min(X_n + 1, M) & \text{w/prob } 1-p \\ 0 & \text{w/prob } p \end{cases}$$

Alternatively

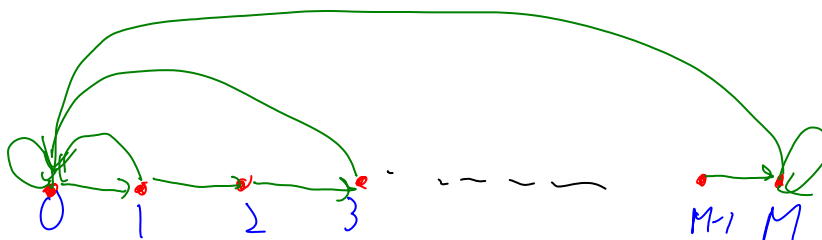
$$X_{n+1} = \min(X_n + 1, M) Z_n$$

where  $Z_n = \begin{cases} 1 & \text{w/prob } 1-p \\ 0 & \text{w/prob } p \end{cases}$   
*iid rvs*

Probability transition matrix:

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & \dots & M \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ M \end{matrix} & \begin{pmatrix} 0 & 1-p & 0 & \dots & 0 \\ p & 0 & 1-p & \dots & 0 \\ p & 0 & 0 & 1-p & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & 0 & 0 & \dots & 0 \\ \vdots & 0 & 0 & \dots & 0 \\ p & 0 & 0 & \dots & 0 \\ & & & & 1-p \\ & & & & 1-p \end{pmatrix} \end{matrix}$$

Let's look for a stationary distribution for this Markov chain. How do we know we will find one? Let's check whether it's irreducible and/or aperiodic to see what to expect.



This Markov chain is irreducible because any two states communicate. Starting from any state  $i$ , one has the possibility to go to state  $0$ , from which it is subsequently possible to reach state  $j$  in  $j$  more steps.

The Markov chain is aperiodic because state  $0$  (or  $M$ ) can return to themselves in one epoch, so they have period  $1$ , and the whole chain is one class which has the same period for each state.

So this guarantees us the existence and uniqueness of the stationary distribution and that it serves as a limit distribution for the system.

Now we calculate the stationary distribution. Let's just use the straight definition and linear algebra so we can get analytic expression. Detailed balance won't work here.

$$\vec{\pi} \cdot P = \vec{\pi}$$

$$(\pi_0 \pi_1 \dots \pi_M) \begin{pmatrix} p & 1-p & 0 & \dots & 0 \\ 0 & 0 & 1-p & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1-p \\ p & 0 & \dots & 0 & 1-p \end{pmatrix} = (\pi_0 \pi_1 \dots \pi_M)$$

$$\begin{cases} p \left( \sum_{j=0}^M \pi_j \right) = \pi_0 \\ \pi_{j-1} (1-p) = \pi_j \quad \text{for } 1 \leq j \leq M-1 \\ (\pi_{M-1} \pi_M) (1-p) = \pi_M \\ \sum_{j=0}^M \pi_j = 1 \Rightarrow \pi_0 = p \Rightarrow \pi_j = p(1-p)^j \quad \text{for } 0 \leq j \leq M-1 \\ \pi_M (1 - (1-p)) = (1-p) \pi_{M-1} = (1-p) p (1-p)^{M-1} \\ \pi_M = (1-p)^M \end{cases}$$

$$\begin{cases} \pi_j = p(1-p)^j \quad \text{for } 0 \leq j \leq M-1 \\ \pi_M = (1-p)^M \end{cases}$$

Now let's use this stationary distribution along with the Law of Large Numbers for Markov chains to compute answers to our questions based on our model.

First question we'll address: What fraction of products are inspected in the long run?

$$I = \frac{\# \text{ inspections}}{\# \text{ products}} = \lim_{n \rightarrow \infty} \frac{n}{\sum_{i=1}^n T_{n,i}}$$

$$\text{where } T_n = \begin{cases} 1 & \text{if } 0 \leq X_n \leq M-1 \\ r & \text{if } X_n = M \end{cases}$$

is the number of products counting from the  $n$ th inspected product up to but not including the  $n+1$  inspected product.

$$\begin{aligned} I &= \left( \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n T_{n,i} \right)^{-1} = \left( \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f(X_{n,i}) \right)^{-1} \\ &= \left( \sum_{j=0}^{M-1} f(j) \pi_j \right)^{-1} \quad (\text{LLN for MC}) \\ &= \left( \sum_{j=0}^{M-1} 1 \cdot p(1-p)^j + r(1-p)^M \right)^{-1} \\ &= \frac{1}{1 + (r-1)(1-p)^M} \end{aligned}$$