The SIMPLEX ALGORITHM FOR LINEAR PROGRAMMING

Simplex is a method for solving linear programming problems.

Standard Form

A linear program is in STANDARD FORM if it is

- a minimization problem
- with equality constraints, and
- all variables are nonnegative.

So, a linear program in standard form looks like

$$\min \quad z(x) = d + c^T x$$

$$\text{subject to} \quad Ax = b$$
$$x \geq 0,$$

where d is a constant,

$$c^T x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$Ax = b$ is the matrix form of a system of linear equalities.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad \cancel{\text{Eg}}$$

Eg. $\begin{aligned} 2x_1 + x_2 - 3x_3 &= 2 \\ x_1 \quad\quad + 5x_3 &= 4 \end{aligned} \longrightarrow A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(A_i = ith row of A, so here $A_1 = [2 \ 1 \ -3]$, $A_2 = [1 \ 0 \ 5]$

We require $A_1 x = b_1$, $A_2 x = b_2$) DON'T NEED THIS NOTATION

Reformulating any Linear Program into Standard Form.

Maximization problems

$$\max \{ c^T x \ \text{ s.t. } Ax = b,\ x \geq 0 \}$$

$$= -\min \{ -c^T x \mid Ax = b,\ x \geq 0 \}.$$

So solve
$$\min\ -c^T x$$
$$Ax = b$$
$$x \geq 0.$$

Eg: $\max \quad x_1 + 2x_2 - x_3$

s.t. constraints

Instead: $\min \quad -x_1 - 2x_2 + x_3$

s.t. constraints.

Inequality constraints

Add slack variables.

Eg: $\min \quad x_1 + 2x_2 - x_3$

$$\text{s.t.} \quad \begin{aligned} -x_1 + x_2 \quad & \leq 2 \\ x_1 + 3x_2 + x_3 & = 4 \\ x_1 - x_2 - x_3 & \geq +2 \\ x_i & \geq 0. \end{aligned}$$

Change things to $\leq$ by multiplying by -1:

$\min \quad x_1 + 2x_2 - x_3$

$$\text{s.t.} \quad \begin{aligned} -x_1 + x_2 \quad & \leq 2 \\ x_1 + 3x_2 + x_3 & = 4 \\ -x_1 + x_2 + x_3 & \leq -2 \end{aligned}$$

probably don't want to change sign if using method of artificial variables.

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Add slack variables:

$$\begin{aligned} \min \quad & x_1 + 2x_2 - x_3 \\ \text{s.t.} \quad & -x_1 + x_2 + x_4 = 2 \\ & x_1 + 3x_2 + x_3 = 4 \\ & -x_1 + x_2 + x_3 + x_5 = -2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

Free Variables

A FREE VARIABLE is one which is unrestricted in sign, so it can be positive or negative (or zero).

Notice:

Any number can be written as the difference of two nonnegative numbers:

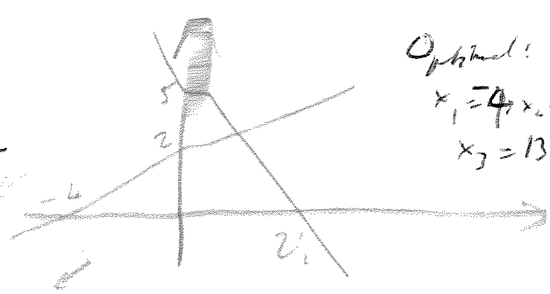
Eg: $5 = 6 - 1 = 5 - 0$
$\quad -3 = 1 - 4 = 0 - 3$

So:

Can replace a free variable x_i by two nonnegative variables x_i', x_i'', and write $x_i' - x_i''$ for x_i in the constraints and objective.

Eg:

$$\begin{aligned} \min \quad & x_1 + x_2 \\ & 2x_1 + x_2 + x_3 = 5 \\ & -x_1 + 2x_2 + x_4 = 4 \\ & x_2 \geq 0, \; x_3 \geq 0, \; x_4 \geq 0 \end{aligned}$$



Optimal:
$x_1 = -4, x_2 = 0,$
$x_3 = 13, x_4 = 0.$

Equivalent to:

$$\begin{aligned} \min \quad & x_1 + x_2' - x_2'' \\ \text{s.t.} \quad & 2x_1 + x_2' - x_2'' + x_3 = 5 \\ & -x_1 + 2x_2' - 2x_2'' + x_4 = 4 \\ & x_1, x_2', x_2'', x_3, x_4 \geq 0. \end{aligned}$$

Equivalent to:

$$\min \quad x_1' - x_1'' + x_2$$

$$\text{s.t.} \quad 2x_1' - 2x_1'' + x_2 + x_3 = 5$$
$$-x_1' + x_1'' + 2x_2 + x_4 = 4$$
$$x_1', x_1'', x_2, x_3, x_4 \geq 0.$$

Optimal: $\underbrace{x_1' = 0, \; x_1'' = 4}_{\text{so } x_1 = x_1' - x_1'' = -4.}, \; x_2 = 0, \; x_3 = 13, \; x_4 = 0$

Infinitely many optimal solutions:

$\underbrace{x_1' = s, \; x_1'' = 4+s}_{\text{still } x_1 = x_1' - x_1'' = -4.}, \; x_2 = 0, x_3 = 13, \; x_4 = 0 \quad$ for $s \geq 0$.

Only basic if $s = 0$.

In general:

$$\min \quad c^T x + d^T y$$
$$\text{s.t.} \quad Ax + By = b$$
$$x \geq 0 \quad y \text{ free}$$

is equivalent to

$$\min \quad c^T x + d^T(u - v)$$
$$\text{s.t.} \quad Ax + B(u-v) = b$$
$$x, u, v \geq 0$$

i.e.,

$$\min \quad c^T x + d^T u - d^T v$$
$$\text{s.t.} \quad Ax + Bu - Bv = b$$
$$x, u, v \geq 0.$$

If had k free variables, new problem has an additional k variables.

An alternative method which only increases the number of variables by one:

Eg: $$\begin{bmatrix}1\\2\\-3\\4\end{bmatrix} = \begin{bmatrix}4\\5\\0\\7\end{bmatrix} - \begin{bmatrix}3\\3\\3\\3\end{bmatrix} = \begin{bmatrix}4\\5\\0\\7\end{bmatrix} - 3\underbrace{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}_{=:\,e}$$

In general: $y = u - we$

u: vector $\quad w$: scalar.

So: $$\begin{aligned} \min\ & c^T x + d^T y \\ \text{s.t. } & Ax + By = b \\ & x \geq 0 \quad y \text{ free} \end{aligned}$$

is equivalent to

$$\begin{aligned} \min\ & c^T x + d^T (u - we) \\ \text{s.t. } & Ax + B(u - we) = b \\ & x \geq 0,\ u \geq 0,\ w \geq 0 \end{aligned}$$

ie,

$$\begin{aligned} \min\ & c^T x + d^T u - (d^T e) w \\ \text{s.t. } & Ax + Bu - (Be) w = b \\ & x \geq 0,\ u \geq 0,\ w \geq 0. \end{aligned}$$

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E.g. $\min \; 2x_1 + 5x_2 - x_3 - x_4$

$$\text{s.t.} \quad \begin{aligned} x_1 + x_2 - x_3 - x_4 &= 10 \\ -2x_1 + 2x_3 - 3x_4 &= 5 \\ x_1 \geq 0,\; x_2, x_3, x_4 &\text{ free.} \end{aligned}$$

~~Equivalent to:~~

~~$\min \; 2x_1 + 5x_2 - x_3 - x_4$~~

Here, $d = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$, so $d^T e = 3$

$B = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & -3 \end{bmatrix}$ so $Be = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$

Thus, equivalent to:

$\min \; 2x_1 + 5x_2' - x_3' - x_4' - 3w$

$$\text{s.t.} \quad \begin{aligned} x_1 + x_2' - x_3' - x_4' + w &= 10 \\ -2x_1 + 2x_3' - 3x_4' + 3w &= 5 \\ x_1, x_2', x_3', x_4', w &\geq 0 \end{aligned}$$

If have solution x_1, x_2', x_3', x_4', w to this standard form problem, get solution to original as:

$$x = \begin{bmatrix} x_1 \\ x_2' - w \\ x_3' - w \\ x_4' - w \end{bmatrix}$$

Converting a problem into standard form

Eg: Oil refinery.

Two types of crude oil, Crude A & Crude B

Make three types of gasoline, Regular, Good, Super, by blending the crude oil.

	Regular	Good	Super	Available
Crude A	.2	.5	.7	200
Crude B	.8	.5	.3	400
Price	10	15	20	

(Simpler than earlier example (p.) because only two types of crude, so unique way of blending for each grade of gasoline)

Let x_R, x_G, x_S be amount of Regular, Good, Super gasoline produced.

Can model as an LP:

$$\text{maximize} \quad 10x_R + 15x_G + 20x_S$$

$$\text{s.t.} \quad .2x_R + .5x_G + .7x_S \le 200 \quad \text{Availability of Crude A}$$

$$.8x_R + .5x_G + .3x_S \le 400 \quad \text{Availability of Crude B}$$

$$x_R, x_G, x_S \ge 0.$$

Want to convert this into standard form. Need

- minimization problem
- equality constraints.

Minimization problem:

Could minimize $-10x_R - 15x_G - 20x_S$. ~~since the maximum of~~ ~~a~~

Equality constraints:

Let x_A = amount of Crude A ~~wa~~ which is unused

x_B = ________ B ________

Then we can express the constraints as

$$\begin{aligned} .2x_R + .5x_G + .7x_S + x_A \quad &= 200 \\ .8x_R + .5x_G + .3x_S \quad + x_B &= 400 \end{aligned}$$

So we get an equivalent problem:

$$\min \quad -10x_R - 15x_G - 20x_S$$

$$\text{s.t.} \quad \begin{aligned} .2x_R + .5x_G + .7x_S + x_A \quad &= 200 \\ .8x_R + .5x_G + .3x_S \quad + x_B &= 400 \\ x_R, x_G, x_S, x_A, x_B &\geq 0. \end{aligned}$$

x_A and x_B are called SLACK VARIABLES.

Here, $A = \begin{bmatrix} .2 & .5 & .7 & 1 & 0 \\ .8 & .5 & .3 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 200 \\ 400 \end{bmatrix}$

$$c^T = \begin{bmatrix} -10 & -15 & -20 & 0 & 0 \end{bmatrix}$$

Notice that here we have $d = 0$.

The Simplex Tableau

(Do [illegible] for the [illegible])

Associated with each linear program in standard form is a tableau used for bookkeeping

$$\begin{array}{c|c|c|} & & x_1 \; \cdots \; x_n \\ \hline z & -d & c^T \\ \hline & b & A \\ \hline \end{array}$$

Simplex tableau

First row represents the equation $z - d = c^T x$. (or $z = d + c^T x$)

The remaining rows represent the equations $b = Ax$, (so they represent $b_i = A_i x$, $i = 1, \ldots, m$.)

For the refinery problem, the simplex tableau is

	x_R	x_G	x_S	x_A	x_B
0	-10	-15	-20	0	0
200	.2	.5	.7	1	0
400	.8	.5	.3	0	1

Tableau T_1

The same linear program can be represented by many different simplex tableaus.

E.g. might want to eliminate x_G from second constraint equation.

escale first.

First equation says: $200 = .2x_R + .5x_G + .7x_S + x_A$

$\Rightarrow \quad x_G = 400 - .4x_R - 1.4x_S - 2x_A$

~~$\Rightarrow \quad x_G = 1000 - 2.5x_G - 3.5x_S - 5x_A$~~

is $\underbrace{400 = .4x_R + x_G + 1.4x_S + 2x_A}$

Use this to rescale the first row.

So second constraint is

$$400 = .8x_R + .5x_G + .3x_S + x_B$$

$$\cancel{= .8(1000 - 2.5x_G - 3.5x_S}$$

$$= .8x_R + .5(400 - .4x_R - 1.4x_S - 2x_A) + .3x_S + x_B$$

$$= .6x_R \quad - .4x_S - x_A + x_B + 200$$

or $\quad 200 = .6x_R \quad - .4x_S - x_A - x_B$

So refinery problem can also be represented by the tableau:

	x_R	x_G	x_S	x_A	x_B
0	-10	-15	-20	0	0
400	.4	1.0	1.4	2	0
200	.6	0	-.4	-1	1

A_{22}

Notice: could have done this by ~~subtracting~~ adding $-\frac{(.5)}{.5}$ × first constraint row to second constraint row

Elementary row operation as in Gaussian elimination.

Do tableau T1 on page OR I 8 first.

Basic Solutions A BASIC SOLUTION to a linear program in standard form is one obtained by fixing just enough variables to zero that the model's equality constraints can be solved uniquely for the remaining variable values.

Eg: In tableau T1:

(A) Fix $x_R, x_G, x_S = 0$. Then take $x_A = 200$, $x_B = 400$.

(B) Fix $x_R, x_S, x_A = 0$. ~~Then take~~

Then need
$$\left.\begin{aligned} .5x_G &= 200 \\ .5x_G + x_B &= 400 \end{aligned}\right\} \quad x_G = 400,\ x_B = 200.$$

Note: 2 equations in 2 unknowns, so soln is unique (provided columns are linearly independent)

(C) Fix $x_R, x_A, x_B = 0$.

Then need
$$.5x_G + .7x_S = 200 \quad (1)$$
$$.5x_G + .3x_S = 400 \quad (2)$$

(1) − (2) $\Rightarrow$ $.4x_S = -200$ $\Rightarrow$ $x_S = -500$, $x_G = 1100$

Defn The variables fixed to zero are NONBASIC.
The ones obtained by solving the equalities are BASIC.

Defn A BASIC FEASIBLE SOLUTION is a basic solution whose basic variables are all nonnegative.

Note: The basic feasible solutions of a linear program in standard form are exactly the extreme point solutions of its feasible region. Do example page OR I 13.

They just calculate the bfs, and then they are extreme.

Adjacent extreme points & edges

~~Def~~ The extreme points of an LP feasible space are ADJACENT if they are determined by active constraint sets differing in only one element.

Defn An EDGE of the feasible region for a linear program is a 1-dimensional set of feasible points along a line determined by a collection of active constraints.

(Illustrate with example on page ORI 13.)

Note A basic solution exists only if the columns of the equality constraints corresponding to basic variables ~~fo~~ are linearly independent.

Can look for largest possible sets of linearly independent columns: such ~~these~~ a set gives a basis for $\mathbb{R}^m$ (A is $m \times n$).

Canonical Form

Eg:

$$
\begin{array}{c|c|ccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
z & -\cancel{8}3 & 0 & 0 & -3 & 0 & 1 \\
\hline
 & 2 & 0 & 1 & 1 & 0 & -1 \\
 & 1 & 1 & 0 & -2 & 0 & 0 \\
 & 7 & 0 & 0 & 3 & 1 & 4
\end{array}
$$

From the form of the simplex tableau, we can easily write down a feasible solution: let $x_3 = 0$, $x_5 = 0$, ~~take~~ then the constraint equations yield $x_2 = 2$, $x_1 = 1$, $x_4 = 7$.

So $x^0 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 7 \\ 0 \end{bmatrix}$ $\quad z(x^0) = 3\cancel{8} - 3x_3 + x_5 = \cancel{8}3$

Definition A simplex tableau

$$
\begin{array}{c|c}
 & x_1 \; \dots \; x_n \\
\hline
-d & c^T \\
\hline
b & A
\end{array}
$$

is in canonical form if

- $b_i \geq 0$ for $i = 1, \dots, m$
- the matrix A contains the m identity columns of the $m \times m$ identity matrix I_m
- the objective function coefficients corresponding to those m identity columns are zero

Example is in canonical form because:

- b_1, b_2, b_3 are ≥ 0
- the second, first and fourth columns of A give $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- the corresponding objective function coefficients c_2, c_1, c_4 are zero.

Variables corresponding to the m identity columns are BASIC VARIABLES - x_2, x_1, x_4

Remaining variables are NONBASIC VARIABLES - x_3, x_5.

Simplex Directions

Def SIMPLEX DIRECTIONS are constructed by increasing a single nonbasic variable, leaving the other nonbasics unchanged, and computing the (unique) corresponding changes in the basic variables necessary to preserve equality constraints.

Matrix representation:

	x_R	x_G	x_S	x_A	x_B	
min c	-10	-15	-20	0	0	b
	.2	.5	.7	1	0	200
A	.8	.5	.3	0	1	400
	N	N	N	B	B	
$x^{(0)}$	0	0	0	200	400	

Eg: Moving from T1 to T2:

∃ Have bfs with basic variables x_A, x_B

Increase nonbasic variable x_G.

Keep $x_R = x_S = 0$.

So need:
$$200 = .5x_G + x_A$$
$$400 = .5x_G + x_B$$

Currently $x_G = 0$, $x_A = 200$, $x_B = 400$.

Look for changes $\Delta x_G, \Delta x_A, \Delta x_B$.

Set $\Delta x_G = 1$ (this is a direction, so it will be scaled by a steplength)

Need:
$$0 = .5\Delta x_G + \Delta x_A$$
$$0 = .5\Delta x_G + \Delta x_B$$

So
$$\Delta x_A = -0.5$$
$$\Delta x_B = -0.5.$$

New point: $x \leftarrow x + \alpha\Delta x$

$x_R = x_S = 0$, $x_G = \alpha\Delta x_G$, $x_A = 200 + \alpha\Delta x_A$ $x_B = 400 + \alpha\Delta x_B$

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Take $\alpha = 400$:

$x_G = 400, \quad x_R = 0, \quad x_B = 200$

Compare with Tableau T2.

Reading for T2:

repeat T1, with extra lines:

	x_R	x_G	x_S	x_A	x_B	
min c	-10	-15	-20	0	0	b
A	$.2$	$.5$	$.7$	1	0	200
	$.8$	$.5$	$.3$	0	1	400
$t=1$	N	B	N	N	B	
$x^{(1)}$	0	400	0	0	200	
Δx for x_R: Solve $.2x_R + .5x_G = 0$, $.8x_R + .5x_G + x_B = 0$						
	1	$-.4$	0	0	$-.6$	$\bar{c}_R = -4$
Δx for other nonbasic						
min ratio						

Improving simplex directions, reduced cost

~~Reduc~~ REDUCED COST $\bar{c}_j$ associated with nonbasic variable x_j is

$$\bar{c}_j = c^T \Delta x$$

Next: x_R enters (frac...

Next: x_S enters, x_G leaves.

(integer, $x_R = 440$, $x_S = 160$

Note The simplex direction increasing nonbasic x_j is improving for a maximize linear program if the corresponding reduced cost $\bar{c}_j > 0$, and for a minimize linear program if $\bar{c}_j < 0$.

Eg: $\Delta x = (0, 1, 0, -\frac{1}{2}, -\frac{1}{2})$ in Tableau T1

$\bar{c}_G = -15$.

Note in Tableau T2: $c^T \Delta x = -15$ again.

Can also eliminate x_G from the objective:

Add $-\frac{(-15)}{\cancel{5}} \times$ first constraint row to objective row:

		x_R	x_G	x_S	x_A	x_B
z	$\cancel{-}$ 6000	-4	0	1	30	0
	400	.4	1.0	1.4	2	0
	200	.6	0	.4	-1	1

Tableau T_2

Note that we can get a feasible solution easily from this tableau:

$$x_R = x_S = x_A = 0 \quad , \quad x_G = 400, \; x_B = 200.$$

This solution has value -6000, since

$$z = -6000 \cancel{+} - 4x_R + 0x_G + x_S + 30x_A + x_B .$$

(Value in maximization problem is 6000)

Pivoting on a simplex tableau

The sequence of elementary row operations performed on Tableau T_1 to obtain Tableau T_2 is called a PIVOT.

To perform a pivot:

(1) Select a nonzero entry a_{hk} in row h and column k of the matrix A.

pivot position — pivot row — pivot column

(2) Multiply the pivot row by $1/a_{hk}$ so that the entry in the pivot position equals 1.

(3) Use elementary row operations to make all other tableau entries in the pivot column equal to zero.

Simplex algorithm: perform a sequence of pivots.

The feasible solution obtained by setting the nonbasic variables equal to zero and using the constraint equations to solve for the basic variables is the BASIC FEASIBLE SOLUTION associated with the tableau.

BASIC SEQUENCE: $(2,1,4)$. $c^T x = 0$ since $c_i = 0$ for basic vars
Thus, $z = -d$.

Tableaus corresponding to feasible linear programs can be put into canonical form by using an appropriate sequence of pivots. — see ~~next time~~ later.

<u>nding a better ~~bf~~ basic feasible solution.</u>

In example, $z(x) = 3 - 3x_3 + x_5$.

Want to decrease z. So try to increase x_3 (currently 0).

If $x_3 = t > 0$ and $x_5 = 0$ then basic variables have values:

$$x_2 = 2 - t \qquad \text{and} \qquad z(x(t)) = 3 - 3t$$
$$x_1 = 1 + 2t$$
$$x_4 = 7 - 3t$$

Want t as large as possible.

We are satisfying $Ax = b$ for any t.

What about nonnegativity?
Need $2 - t \geq 0$, $1 + 2t \geq 0$, $7 - 3t \geq 0$
OK provided $t \leq 2$.

So take $t = 2$:
Get $x_2 = 0$, $x_1 = 3$, $x_4 = 1$, $x_3 = 2$, $(x_5 = 0)$.

Is this a basic feasible solution?

		x_1	x_2	x_3	x_4	x_5
z	-3	0	0	-3	0	1
	2	0	1	1	0	-1
	1	1	0	-2	0	0
	7	0	0	3	1	4

Pivot on circled entry: x_3 column, x_2 row.

Get:

		x_1	x_2	x_3	x_4	x_5
z	3	0	3	0	0	-2
	2	0	1	1	0	-1
1	4	1	2	0	0	-2
	1	0	-3	0	1	7

Canonical with basis sequence $(3, 1, 4)$

Why did we choose this pivot element?

$$x(t) = \begin{bmatrix} 1+2t \\ 2-t \\ t \\ 7-3t \\ 0 \end{bmatrix} \geq 0 \quad \text{if and only if } \cancel{t \geq 0} \quad t \leq 2 \text{ and } t \leq 7/3 \quad (\text{and } t \geq 0)$$

So $x(t) \geq 0$ if and only if $t \leq \min\left\{\frac{2}{1}, \frac{7}{3}\right\}$

Notice that this is the minimum of $\frac{b_h}{a_{hk}}$ for h with $a_{hk} > 0$

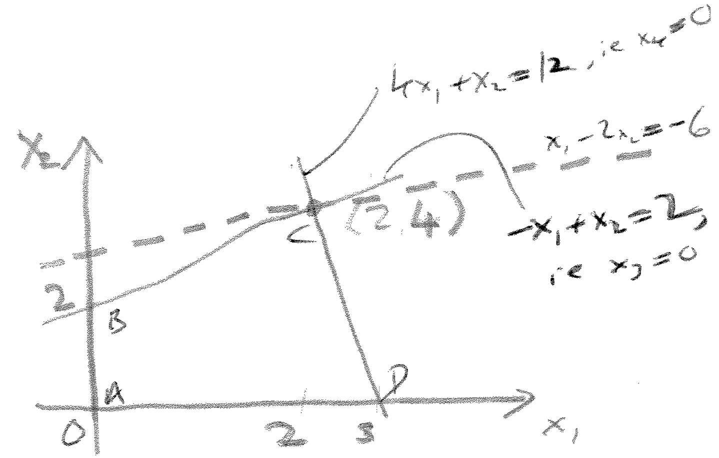
Minimum ratio rule: Choose row h with $\frac{b_h}{a_{hk}} = \min\left\{\frac{b_h}{a_{hk}} : a_{hk} > 0\right\}$.

Maintains $b \geq 0$.

Simplex Rule

- select the pivot column as any column with $c_k < 0$
- Given k, select the pivot row h as a minimum ratio row associated

The Geometry of a Pivot



Eg:

$$\begin{aligned} \min \quad & x_1 - 2x_2 \\ \text{st.} \quad & -x_1 + x_2 \le 2 \\ & 4x_1 + x_2 \le 12 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Introduce slacks:

$$\begin{aligned} \min \quad & x_1 - 2x_2 \\ & -x_1 + x_2 + x_3 = 2 \\ & 4x_1 + x_2 + x_4 = 12 \\ & x_i \ge 0. \end{aligned}$$

Tableau:

		x_1	x_2 ↓	x_3	x_4
z	0	1	-2	0	0
$\frac{2}{1}$	2	-1	1	1	0
$\frac{12}{1}$	12	4	1	0	1

Canonical form. Bfs: $x_1 = x_2 = 0$ $x_3 = 2$, $x_4 = 12$ (A)
Value 0

Pivot:

		x_1 ↓	x_2	x_3	x_4
z	4	-1	0	2	0
—	2	-1	1	1	0
$\frac{10}{5}$	10	5	0	-1	1

Canonical form.
Bfs: $x_1 = x_3 = 0$
$x_2 = 2$, $x_4 = 10$ (B)
Value -4

Pivot:

		x_1	x_2	x_3	x_4
z	6	0	0	$9/5$	$1/5$
	4	0	1	$4/5$	$1/5$
	2	1	0	$-1/5$	$1/5$

Canonical form
Bfs: $x_3 = x_4 = 0$
$x_2 = 4$, $x_1 = 2$ (C)
Value: -6.

Optimal, Unbounded, & Infeasible Forms.

Optimal Form

x^* MINIMIZING VECTOR if $z(x) \geq z(x^*)$ for all feasible x.

Eg:

		x_1	x_2	x_3	x_4
z	6	0	0	$\frac{9}{5}$	$\frac{1}{5}$
	4	0	1	$\frac{4}{5}$	$\frac{1}{5}$
	2	1	0	$-\frac{1}{5}$	$\frac{1}{5}$

Here, $z(x) = -6 + \frac{9}{5} x_3 + \frac{1}{5} x_4$

Current basic feasible solution: $\hat{x}_2 = 4, \hat{x}_1 = 2, (\hat{x}_3 = \hat{x}_4 = 0)$

Any feasible vector has $x_3 \geq 0$ and $x_4 \geq 0$.

So any feasible solution has $z(x) = -6 + \frac{9}{5} x_3 + \frac{1}{5} x_4 \geq -6$.

$$= z(\hat{x}).$$

Thus, $\hat{x}$ is a minimizing point.

NB!

A tableau in canonical form is in

OPTIMAL FORM if $c_j \geq 0$ for each j.

Unbounded form

Eg:

		x_1	x_2	x_3	x_4	x_5
z	4	0	-1	2	0	0
	2	0	-2	1	0	1
	3	1	-6	3	0	0
	1	0	0	-2	1	0

Basic feasible solution: $x_5 = 2,\ x_1 = 3,\ x_4 = 1 \qquad x_2 = x_3 = 0$

$z(x) = -4 - x_2 + x_3.$

Try to increase x_2.

Minimum ratio test: look at $\left\{ \frac{b_i}{a_{i2}} : a_{i2} \geq 0 \right\}$ – empty set!

What does this mean?

Let $x_2 = t$. Then $x(t) = \begin{bmatrix} 3+6t \\ 0+t \\ 0 \\ 1 \\ 2+2t \end{bmatrix} \geq 0$ for all $t \geq 0$.

So no limit on t.

Ie, $x(t)$ is feasible for all $t \geq 0$.

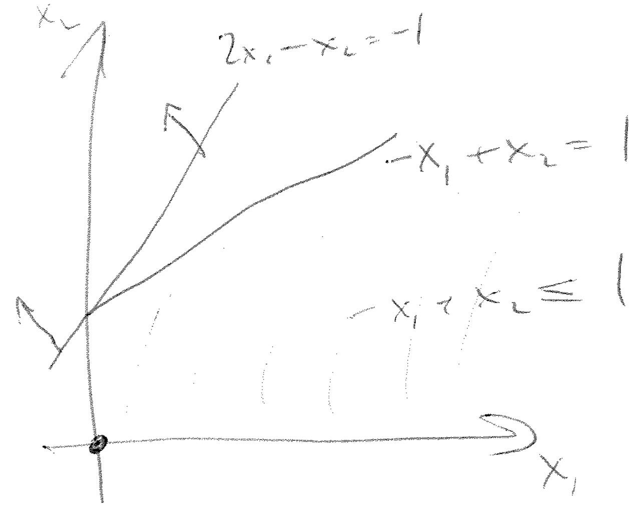
Now, $z(x(t)) = -4 - t \to -\infty$ as $t \to +\infty$.

So <u>optimal value is unbounded</u>. – NB not just feasible region

Tableau in <u>unbounded form</u> if $c_k < 0$ for some k, and $a_{ik} \leq 0$ for each i.

An example with unbounded feasible region, and finite optimal value ORI 150

$$\min \; +2x_1 - x_2$$
$$\text{s.t.} \; -x_1 + x_2 \le 1$$
$$x_i \ge 0.$$



$$\min \; +2x_1 - x_2 + 0x_3$$
$$\text{s.t.} \; -x_1 + x_2 + x_3 = 1$$
$$x_i \ge 0$$

0	2	-1	0
1	-1	1	1

Initial sol: $x_1 = x_2 = 0, \; x_3 = 1.$

1	1	0	1
1	-1	1	1

Optimal form.
So optimal sol is $x_2 = 1$
$x_1 = x_3 = 0$
$z = -1.$

An obj fn with infinite optimal value: $\min \; -x_1 - x_2.$

Two Infeasible Forms

Need $Ax = b$, $x \geq 0$.

May have no solution to $Ax = b$:

		x_1	x_2	x_3	x_4
	2	1	2	-2	3
	2	1	1	-1	2
	1	-1	-1	1	-2

↓

		x_1	x_2	x_3	x_4
z	4	-1	0	0	-1
	3	0	0	0	0
	1	-1	-1	1	-2

First constraint says: $0x_1 + 0x_2 + 0x_3 + 0x_4 = 3$ IMPOSSIBLE.

Infeasible form 1: if $b_i \neq 0$ for some i and $a_{ij} = 0$ for each j.

May have solutions to $Ax = b$, but no nonnegative solutions to $Ax = b$:

		x_1	x_2	x_3	x_4
	2	1	2	-2	3
	2	4	1	3	0
	-1	2	0	3	1

Second constraint says: $2x_1 + 3x_3 + x_4 = -1$: IMPOSSIBLE ~~for~~ $x \geq 0$.

Infeasible form 2: if $b_i < 0$ for some i and $a_{ij} \geq 0$ for each j.

Solving Linear Programs in Canonical Form.

Pivoting to Optimal Form.

(Best improvement rule) — Shows that as the better in some situations

An LP in canonical form is either

- in optimal form
- in unbounded form
- c_k is negative for some k and at least one $a_{ik} > 0$.

STOP in first two cases. In last case, pivot, and eventually (after several pivots) end up in optimal or unbounded form.

Eg:

		x_1	x_2	x_3	x_4	x_5
z	3	−4	~~3~~ −1	1	0	0
	20	−1	4	1	0	1
	1	1	−1	1	1	0

Bfs: $x_5 = 20$, $x_4 = 1$
$(x_1 = x_2 = x_3 = 0)$
Value: -3

		x_1	x_2	x_3	x_4	x_5
z	7	0	−5	5	4	0
	21	0	3	2	1	1
	1	1	−1	1	1	0

Bfs: $x_5 = 21$, $x_1 = 1$
$(x_2 = x_3 = x_4 = 0)$
Value $= -7$

42	0	0	$25/3$	$13/3$	$5/3$
7	0	1	$2/3$	$1/3$	$1/3$
8	1	0	$5/3$	$4/3$	$1/3$

Bfs: $x_2 = 7$, $x_1 = 8$
$(x_3 = x_4 = x_5 = 0)$
Value: -42

Optimal.

In first tableau, have a choice for the pivot column – either x_1 or x_2. We chose the column with the MOST NEGATIVE c_j. Gives greatest initial decrease in objective function value – if $x_j = \varepsilon$, change in objective $= c_j \varepsilon$.
May not give largest decrease in $z(x)$. Eg, if pivot on second column:

		x_1	x_2	x_3	x_4	x_5
z	3	-4	-1	1	0	0
$20/4$	20	-1	4	1	0	1
—	1	1	-1	1	1	0

		x_1	x_2	x_3	x_4	x_5
z	8	$-17/4$	0	$5/4$	0	$1/4$
	5	$-1/4$	1	$1/4$	0	$1/4$
	6	$3/4$	0	$5/4$	1	$1/4$

BFS: $x_2 = 5$, $x_4 = 6$
($x_1 = x_3 = x_5 = 0$)
Value $= -8$.

Improvement is 5 (improvement with x_1 as pivot column is 4)

Can implement other pivot rules: Eg, best improvement.
These can reduce the total number of pivots, (by up to ~20% (CPLEX))
but each pivot is more expensive.