7.411/92.473 OPERATIONS RESEARCH I

Concerned with LINEAR PROGRAMMING.

Will motivate by considering an example.

Oakwood Furniture Company

Company has 12.5 units of wood on hand

Table requires 2 units of wood

Chair requires 1 unit of wood.

Distributor will pay \$20 per table

\$15 per chair

Distributor will not accept more than eight chairs.

Distributor wants at least twice as many ~~tables~~ chairs as ~~chairs~~ tables.

How many tables & chairs should the company produce to maximize its revenue?

Decision variables:

x_1 = number of tables produced
x_2 = number of chairs produced

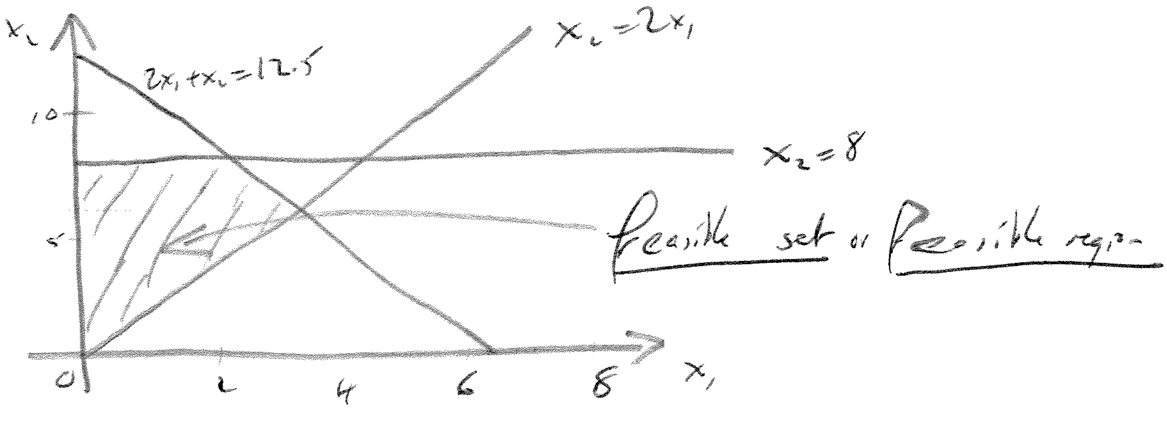
A plan to manufacture this number is a production program.

<u>Income</u> z that would be generated by selling x_1 tables and x_2 chairs would be $z = 20x_1 + 15x_2$

Have various restrictions:

$$\left.\begin{array}{ll} \text{At most eight chairs:} & x_2 \le 8 \\ \text{At least two chairs for each table:} & x_2 \ge 2x_1 \\ \text{Limit on amount of wood available:} & 2x_1 + x_2 \le 12.5 \\ \text{Need nonnegative amount of chairs \& tables:} & x_1 \ge 0, \quad x_2 \ge 0. \end{array}\right\} \underline{\text{Constraints}}$$

A production program that simultaneously satisfies all the constraints is <u>feasible</u>.

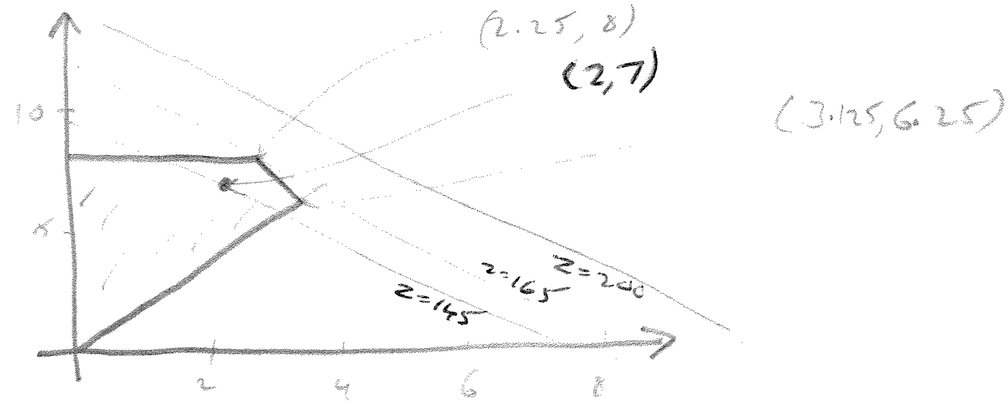


Eg $x_1=2$, $x_2=7$ is feasible. Yields revenue $20.2+15.7$ \$145

Objective of Oakwood is to find the feasible solution with the largest revenue.

The function $z = 20x_1 + 15x_2$ is the <u>objective function</u>.

Look at contours of objective function.



Objective function contour that intersects the feasible region at the highest value is $z = 165$. Have one feasible solution with that value: $x_1 = 2.25$, $x_2 = 8$.

This is the <u>optimal solution</u>. The <u>optimal value</u> is 165.

~~Assumption of continuity~~

Notice that the optimal solution is ~~an extreme point~~ a corner point or <u>extreme point</u>.

The <u>simplex algorithm</u> systematically moves from one corner point to another until it finds the optimal solution.

<u>Assumption of continuity</u>

Optimal solution is 2.25 tables, 8 chairs.

May not want ~~25~~ $\frac{1}{4}$ of a table. So could use <u>integer programming</u> to find best integer solution.

Best integer solution is $x_1 = 2$, $x_2 = 8$, with $z = \$160$.

It may be that fractional values are meaningful, eg.

If the furniture company was measuring in units of 1000, then the solution $x_1 = 2.25$, $x_2 = 8$ corresponds to 2250 tables and 8000 chairs.

Sensitivity of the optimal solution

If the objective function or constraints are changed slightly, what happens to the optimal solution?

Don't necessarily have to solve the problem from scratch – can often use information about the optimal solution.

Algebraic statement of linear programming problems

Oakwood problem:

$$\max \quad z = 20x_1 + 15x_2 \qquad \text{| Objective function.}$$

$$\text{s.t.} \quad \left.\begin{aligned} x_2 &\leq 8 \\ 2x_1 - x_2 &\leq 0 \\ 2x_1 + x_2 &\leq 12.5 \\ x_1, x_2 &\geq 0 \end{aligned}\right\} \text{ Constraints}$$

This is a <u>linear program</u>.

General form:

maximize or minimize a linear function of several variables

subject to linear inequality constraints and/or linear equality constraints.

Diet problem.

Find cheapest combination of foods that will satisfy all the daily nutritional requirements of the person.

See handout from Argonne web site.

Transportation problems

Get goods from warehouses to stores.

Which warehouse supplies which store?

Constraints on capacity of warehouses, demands of stores.

Cost of shipping between each pair.

Eg ~~Blending problem~~

Have ~~five~~ three types of raw oils:

	Octane level.	Cost of raw material
OIL 1	80	.75
OIL 2	90	1.00
OIL 3.	95	1.10

Blend to form two gasolines: premium, regular.

Premium has octane level $\geq$ 92 Sell for \$1.20 / gallon

Regular has octane level $\geq$ 84. Sell for \$1.00 / gallon.

Octane concentration blends linearly.

Refinery can only blend 5000 gallons / day.

So: ~~max $92x_p + 84x_R - 75$~~

$$\max \; 120x_p + 100x_R - 75(x_{p1} + x_{R1}) - 100(x_{p2} + x_{R2}) - 110(x_{p3} + x_{R3})$$

$$\text{s.t.} \quad x_p = x_{p1} + x_{p2} + x_{p3}$$

$$x_R = x_{R1} + x_{R2} + x_{R3}$$

$$x_p + x_R \leq 5000$$

$$80x_{p1} + 90x_{p2} + 95x_{p3} \geq 92x_p$$

$$80x_{R1} + 90x_{R2} + 95x_{R3} \geq 84x_R$$

$$x_{\cdot} \geq 0.$$

Real world examples (use extensions of LP)

1. SABRE crew scheduling, and other scheduling operations.

   Saves hundreds of millions of dollars / year for AA.

2. SABRE yield management:

   Price airline seats on basis of time ahead of purchase.

   Saves AA $1 billion per year.

3. Schedule basketball games: ACC. (p. 22, ~~ED~~ OR/MS Today, April 1998)

   Various constraints:
   - Each team has final game at home every other year
   - Good TV games each week.
   - Players have enough time between games for studies and rest.
   - Schedule convenient for fans.

A shift scheduling example.

Troy Medical is scheduling its resident doctors.

It requires them to work 12 hour shifts, starting at either midnight, ~~B~~ 6am, noon or 6pm.

On average, they need 17 doctors between midnight & 6am
15 ~~24~~ doctors between 6am & noon
20 ~~3~~ doctors between noon & 6pm
31 doctors between 6pm & midnight.

How many doctors do they need to make sure they cover the demand?

Let x_1 = # doctors starting at midnight
x_2 = # doctors starting at 6am
x_3 = # doctors starting at noon
x_4 = # doctors starting at 6pm.

min $x_1 + x_2 + x_3 + x_4$

s.t.

	$x_1 + x_4$	≥ 17	cover midnight – 6am
①	$x_1 + x_2$	$\geq \cancel{24}\ 15$	cover 6am – noon
	$x_2 + x_3$	≥ 20	cover noon – 6pm
②	$x_3 + x_4$	≥ 31	cover 6pm – midnight

$x_1, x_2, x_3, x_4 \geq 0.$

Optimal soln: $x_1 = 15$, $x_2 = 0$, $x_3 = 29$, $x_4 = 2$. Adding ① & ② verifies this is optimal.

~~Opt soln: $x_4 = 31$, $x_2 = 24$ or $x_1 = 24$, $x_3 = 31$.~~

~~Since ① + ② implies $x_1 + x_2 + x_3 + x_4 \geq 24 + 31 = 55$.~~

Possible additional constraints: $x_i \geq 5$, to ensure continuity in service.
(Otherwise, everyone clocks off at noon in the optimal soln.)

A Time-Phased Inventory Model.

Troy Bookstore ~~is selling~~ sells pencils in the Fall semester.

They break the semester into four months:

1. Aug 20 – Sep 20
2. Sep 20 – Oct 20
3. Oct 20 – Nov 20
4. Nov 20 – Dec 20.

They ~~know the~~ have forecast demand as
10000, 2000, 3000, 6000 for the four months

They have 8000 pencils on hand.

They can purchase up to 4000 pencils per month, delivered on the first day of the month

Any pencil not sold in the month it is bought costs .02 ¢ to store.

What should their purchase schedule be to minimize storage costs?

x_i = amount bought in period i
s_i = amount stored at beginning of period i
d_i = demand in period i

$$\min \; 0.02 \sum_{i=2}^{4} s_i$$

$$\text{s.t.} \quad 8000 + x_1 = 10000 + s_2$$
$$s_i + x_i = d_i + s_{i+1} \quad i = 2,3,4.$$
$$x_i \le 4000 \quad i = 1, \dots, 4$$
$$x_i, s_i \ge 0.$$

Optimal:
$x = (2000, 3000, 4000, 4000)$
$s = (0, 1000, 2000)$

(Starting level in period t) + (impact of period t decisions) = starting level in period $t+1$

$$s_t + x_t = d_t + s_{t+1}$$

beginning inv + purchases = demand + ending inventory.

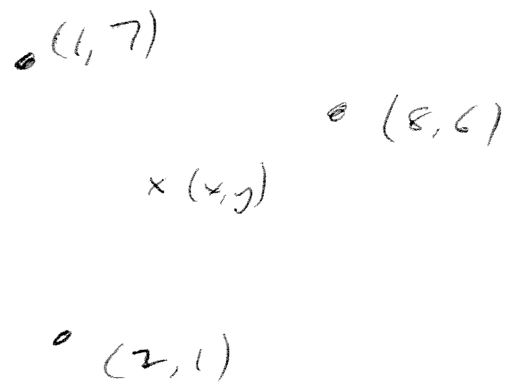
Additional features
1. might want some inventory on hand at end of semester.
2. Pricing differences for purchase cost
3. Stochastic demand

Handling Min-Max & Absolute Values

Want to minimize the maximum distance to three points.

Measure distance using L_1-norm: x-displacement + y-displacement.

Eg: parents want to ~~to~~ minimize the maximum distance to any of their children.



Distance to $(1,7)$: $|x-1|+|y-7|$

Distance to $(8,6)$: $|x-8|+|y-6|$

Distance to $(2,1)$: $|x-2|+|y-1|$

$$\min \max \{ |x-1|+|y-7|, |x-8|+|y-6|, |x-2|+|y-1| \}$$

Nonlinear: min max and absolute values.

Handling min max:

$$\min \quad f$$

$$\text{s.t.} \quad f \geq |x-1|+|y-7|$$
$$f \geq |x-8|+|y-6|$$
$$f \geq |x-2|+|y-1|.$$

Handling absolute values:

Introduce new variables:

$$x-1 = u_{x1} - v_{x1}, \quad u_{x1} \geq 0, \; v_{x1} \geq 0$$

Eg: $x=3$: take $u_{x1}=2, v_{x1}=0$
$x=0$: take $u_{x1}=0, v_{x1}=1$

Then $|x-1| = u_{x1} + v_{x1}$, provided one of these is negative.

$$y-7 = u_{y7} - v_{y7}, \quad u_{y7} \geq 0, \; v_{y7} \geq 0$$

$$|y-7| = u_{y7} + v_{y7}.$$

So require $\quad f \geq u_{x1} + v_{x1} + u_{y7} + v_{y7}$

$$x-1 = u_{x1} - v_{x1}$$
$$y-7 = u_{y7} - v_{y7}$$
$$u_{x1}, v_{x1}, u_{y7}, v_{y7} \geq 0.$$

Note: if u_{x1}, v_{x1} both positive, can do better by reducing them both, leaving x unchanged.

~~Final~~ Final model: $\quad \min f$

s.t. $\quad f \geq u_{x1} + v_{x1} + u_{y7} + v_{y7}$

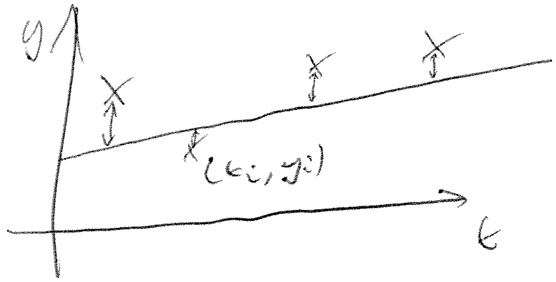
$$f \geq u_{x8} + v_{x8} + u_{y6} + v_{y6}$$

$$f \geq u_{x2} + v_{x2} + u_{y1} + v_{y1}$$

$$x-1 = u_{x1} - v_{x1}$$
$$y-7 = u_{y7} - v_{y7}$$
$$x-8 = u_{x8} - v_{x8}$$
$$y-6 = u_{y6} - v_{y6}$$
$$x-2 = u_{x2} - v_{x2}$$
$$y-1 = u_{y1} - v_{y1}$$

All u's, v's ≥ 0.

Regression



Find line that minimizes the sum of the errors.

Have times t_i and values y_i

Line $y = mt + c$.

Want to find best m, c.

Error at (t_i, y_i) is $|y_i - mt_i - c|$

So: $\min_{m,c} \sum_{i=1}^{n} |y_i - mt_i - c|$

Get rid of absolute values:

$$\min_{m,c,u,v} \sum_{i=1}^{n} u_i + v_i$$

$$\text{s.t.} \quad u_i - v_i = y_i - mt_i - c$$

$$u_i, v_i \geq 0.$$

Baseball playoff elimination

RIOT website:
http://riot.ieor.berkeley.edu/~baseball

Eg:

	W	L
Mudhens	19	15
Lumberjacks	19	15
Jays	16	19
Angels	15	20

Each team plays each other ten times.
Mudhens & Lumberjacks have six games left, all against each other
Jays & Angels have five games left, all against each other.

If Jays win all their remaining games, they would have a record of 21-19.
If Mudhens & Lumberjacks lose all their remaining games, they'd each have records of 19-21
So Jays could come first. (MAGIC NUMBER CALCULATION: eliminate if another team's wins + our losses ≥ 41.)

But either Mudhens or Lumberjacks must win at least 3 of their remaining ~~five~~ six games.
So ~~at least~~ at least one of them will be at least 22-18.
So Jays cannot come first.

Use LP formulation to show Jays can't come first.

parameters Say we have n teams. Team i's current record is W_i wins, L_i losses.
Want to know if team n can still come first.
Can assume they win all their remaining games.

parameter Let p = number of games in a season.

Want to know: is there some selection of results so that our final record of $p - L_n$ wins, L_n losses can win the division?

variables Let W_i^f = final number of wins for team i.
~~parameters~~

parameter Let p_{ij} = number of games remaining between teams i and j.

variables Let g_{ij} = number of these games won by team i
g_{ji} = number of these games won by team j.

So $g_{ij} + g_{ji} = p_{ij}$

Then $W_i^f = W_i + \sum_{\substack{j \neq i \\ j \neq n}} g_{ij}$

Let z = # games by which team n wins the division.
$= \min_{i=1,\dots,n-1} \{ p - L_n - W_i^f \}$

So: $\max \; z$

s.t. $z \leq p - L_n - W_i^f \qquad i = 1, \dots, n-1$

$W_i^f = W_i + \sum_{\substack{j \neq i \\ j \neq n}} g_{ij} \qquad i = 1, \dots, n-1$

$g_{ij} + g_{ji} = p_{ij} \qquad i = 1, \dots, n-1, \; j = 1, \dots, n-1, \; i \neq j.$

g_{ij} integral, nonnegative.

OR I 4-11

This integer program has a special structure:

Can ignore the integrality restriction on g_{ij} — Network flow problem, ~~eliminate~~ g_{ij}^t if don't have z, i.e., if just determining whether a feas soln exists, i.e., a nonnegative See over

If get fractional optimal z, can round it down: this gives the optimal value for the integer program.

If optimal value is < 0, then team n cannot win the division.

For our 4 team example,

$$\max \quad z$$

$$\text{s.t.} \quad z \le 40 - 19 - W_1^f$$

$$z \le 40 - 19 - W_2^f$$

$$z \le 40 - 19 - W_3^f$$

$$W_1^f = 19 + g_{12}$$

$$W_2^f = 19 + g_{21}$$

$$g_{12} + g_{21} = 6$$

g_{ij} integral, nonnegative.

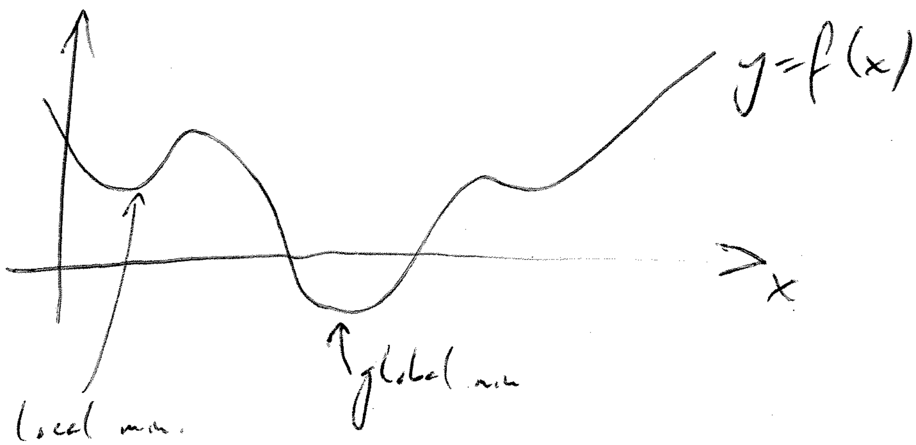
$(W_3^f = 15)$

1 = Mudhens, 2 = Lumberjacks, 3 = Angels

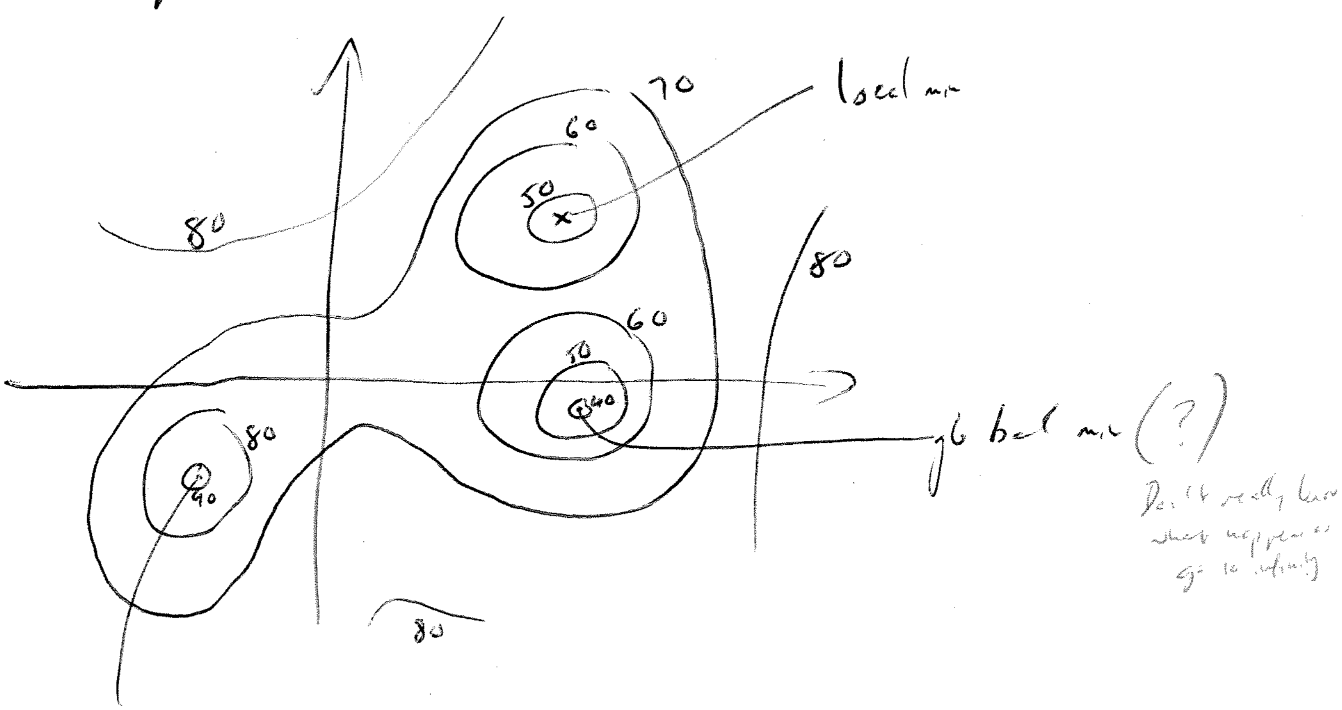
imply either $W_1^f \ge 22$ or $W_2^f \ge 22$, since $W_1^f + W_2^f \ge 44$.

Then $z \le 21 - 22 = -1$. So Jays can't finish first.

Improving Search (Rardin, Chapter 3)

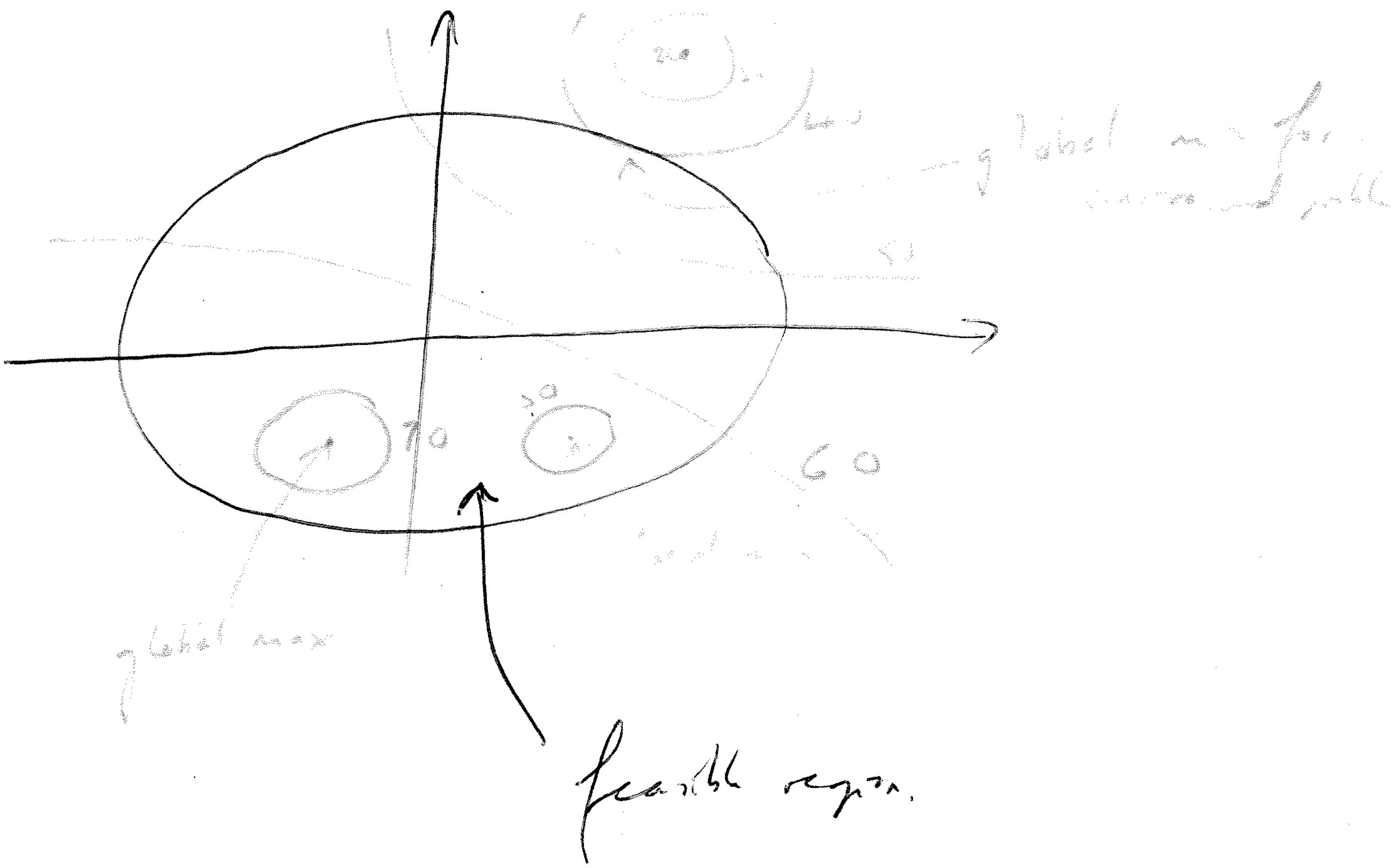


2-D contour plots (unconstrained)

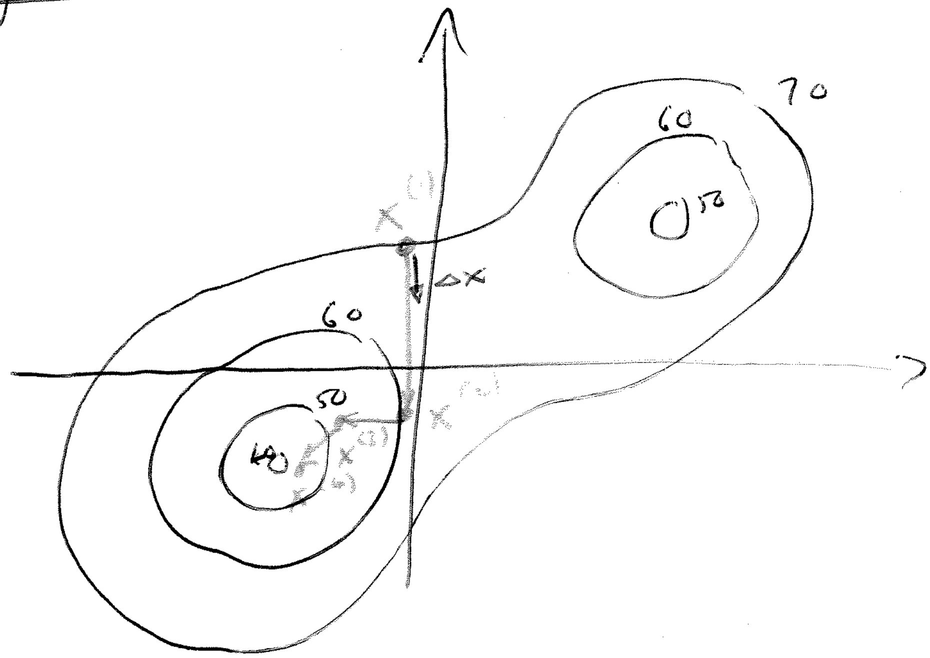


local max.
(not global max, since it appears $f \to \infty$ as $|x| \to \infty$)

2-D contour plot (constrained)



Finding a local min



Advance from current solution $x^{(t)}$ to new solution $x^{(t+1)}$ by choosing a move direction Δx and then a ~~ste~~ step size $\lambda > 0$,

so $x^{(t+1)} = x^{(t)} + \lambda \Delta x$.

In figure, $x^{(2)} = x^{(1)} + 3 \Delta x$.

Vector Δx is an improving direction ~~for~~ at $x^{(t)}$ if the objective function value at $x^{(t)} + \lambda \Delta x$ is superior to that of $x^{(t)}$ for all $\lambda > 0$ sufficiently small.

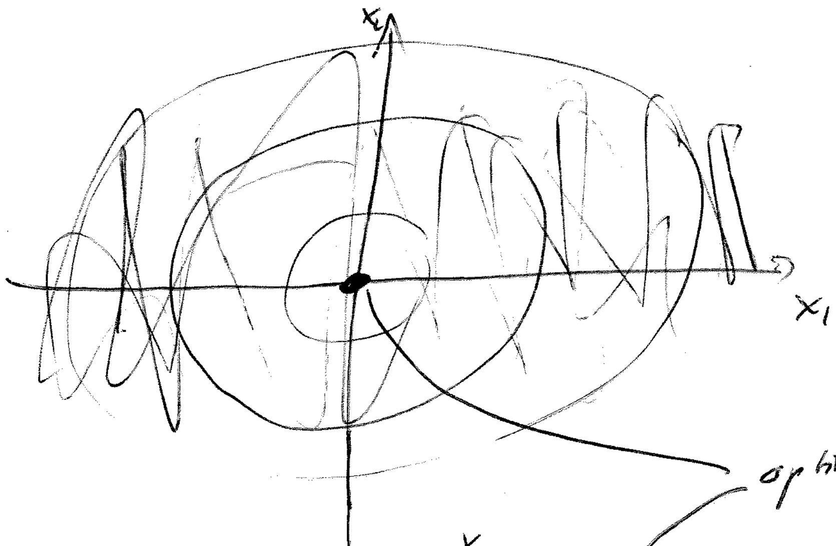
Choosing the direction Δx

Often good to take the negative of the gradient. Since: gradient is direction along which function increases most quickly.

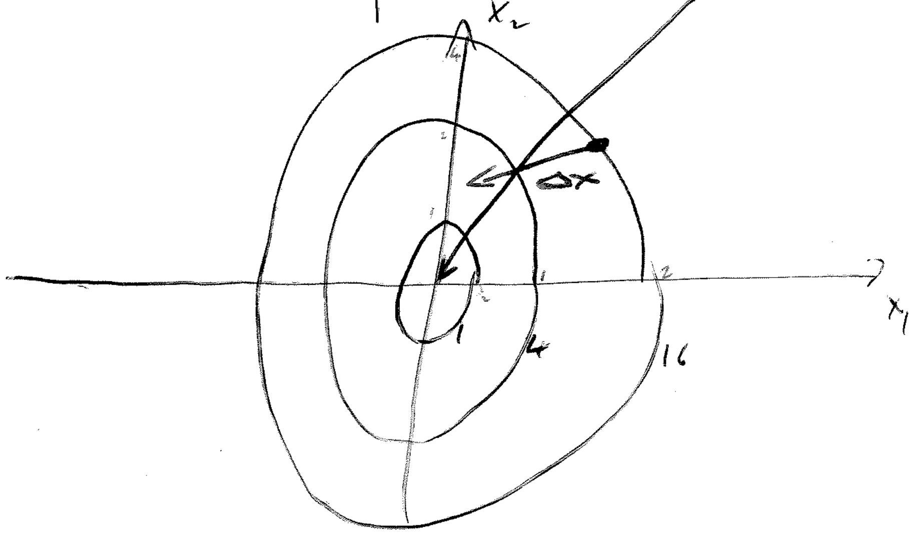
E.g.: $\min 4x_1^2 + x_2^2$

$$f(x) = 4x_1^2 + x_2^2 \qquad \nabla f = \begin{bmatrix} 8x_1 \\ 2x_2 \end{bmatrix}.$$

So take $\Delta x = \begin{bmatrix} -8x_1 \\ -2x_2 \end{bmatrix}.$



optimal solution.



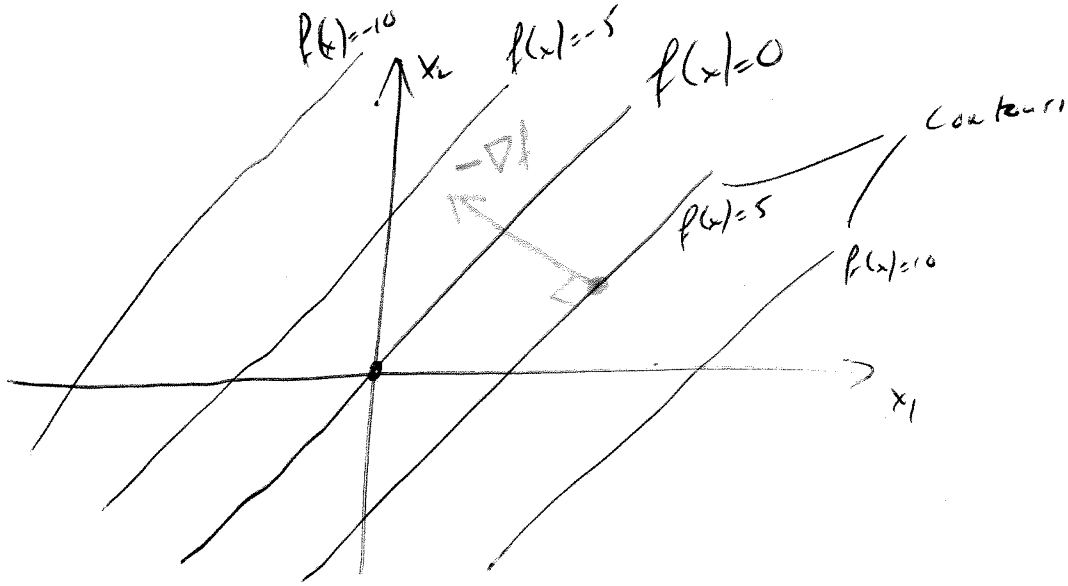
Gradient is orthogonal to contours.

ORI 5 -5

For linear functions:

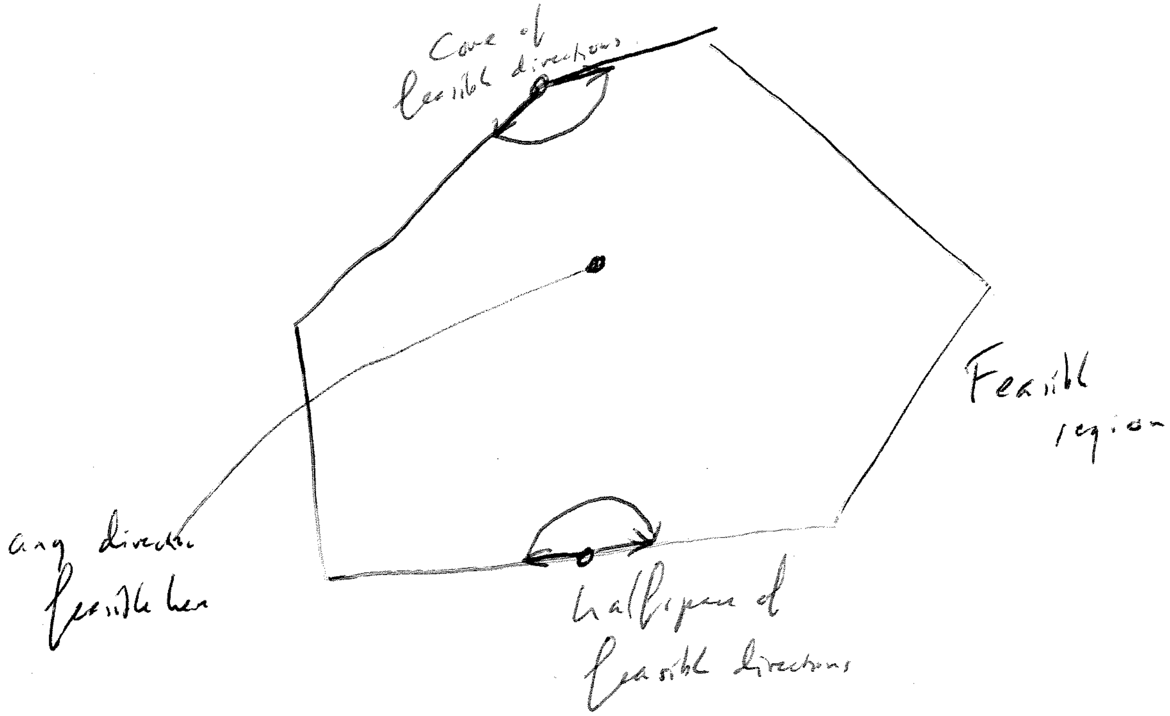
$$f(x) = 5x_1 - 3x_2$$

$$\nabla f = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

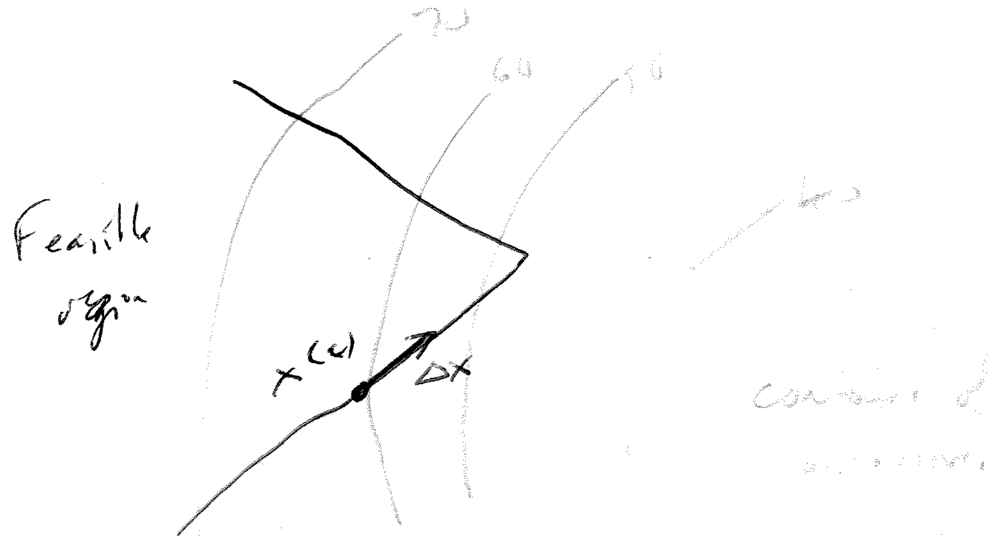


Feasible directions

Vector Δx is a **feasible direction** at the current solution $x^{(t)}$ if point $x^{(t)} + \lambda \Delta x$ violates no model constraint for $\lambda > 0$ sufficiently small.



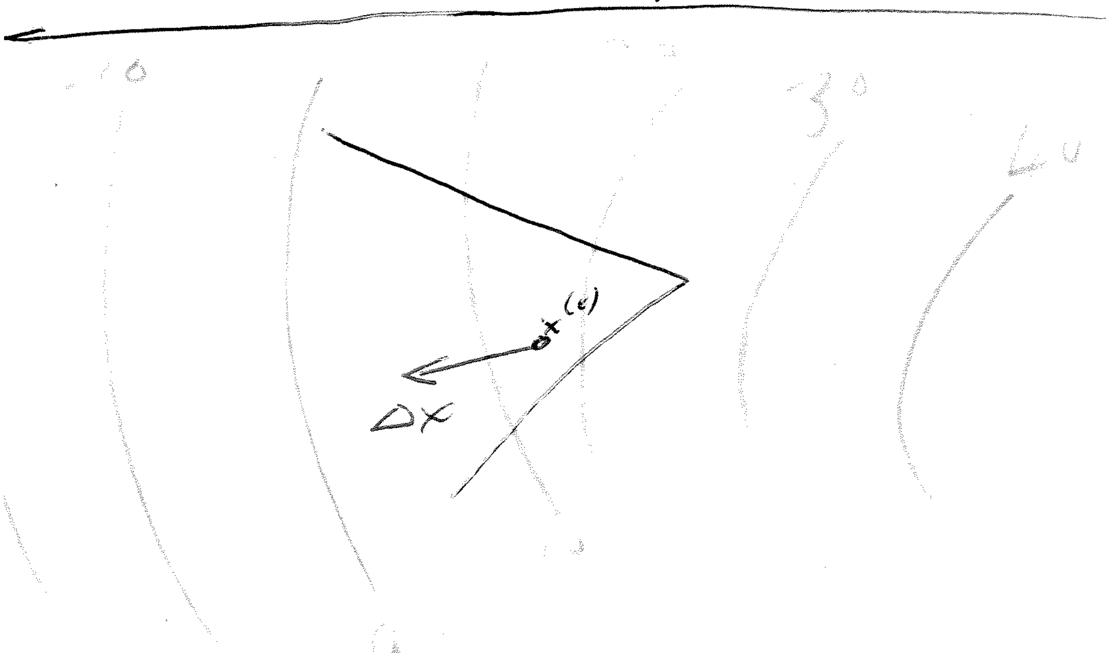
Improving feasible direction.



Can't move in gradient direction.

Can find a direction that improves the objective and maintains feasibility. So $x^{(k)}$ not local maximizer.

Can move until objective no longer improves or until no longer feasible.

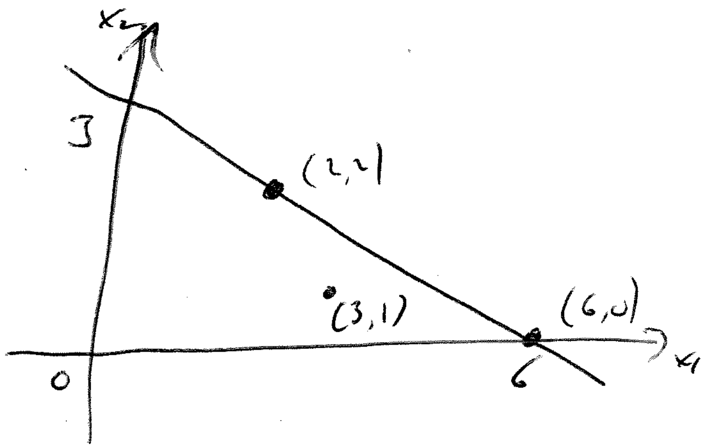


Unbounded:
Can find feasible solution with arbitrarily ~~large~~ good function values.

Active constraint:

A constraint is *active* at $x^{(k)}$ if it holds at equality.

Eg:



$$x_1 + 2x_2 \le 6$$
$$x_1 \ge 0$$
$$x_2 \ge 0.$$

$x_1 + 2x_2 \le 6$ active at $(2, 2)$.

$x_1 + 2x_2 \le 6$, $x_2 \ge 0$ active at $(6, 0)$

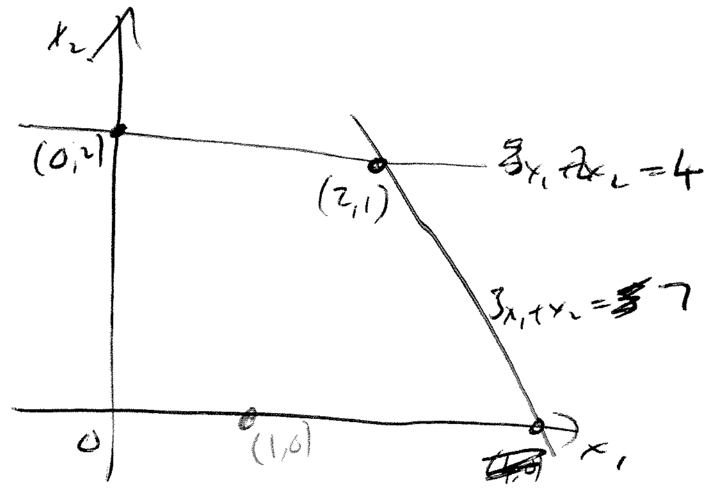
No constraints active at $(3, 1)$.

Equality constraints are always active.

Feasible directions with active constraints

Eg: Have

$$\begin{aligned} 3x_1 + x_2 &\le \cancel{8}\,7 \\ x_1 + 2x_2 &\le 4 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$



Feasible directions $\Delta x = (\Delta x_1, \Delta x_2)$ at $x = (2,1)$ must satisfy:

$$\begin{aligned} 3\Delta x_1 + \Delta x_2 &\le 0 \\ \Delta x_1 + 2\Delta x_2 &\le 0 \end{aligned}$$

since these constraints are active at $x=(2,1)$.

Can have $\Delta x_1 < 0$ and/or $\Delta x_2 < 0$.

At $x = (0,2)$, feasible directions $\Delta x = (\Delta x_1, \Delta x_2)$

Satisfy:

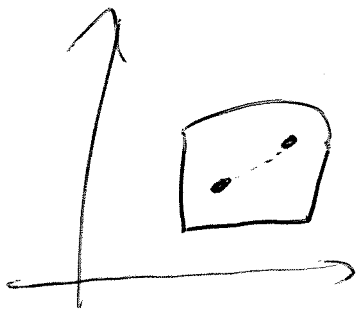
$$\left.\begin{aligned} \Delta x_1 + 2\Delta x_2 &\le 0 \\ \Delta x_1 &\ge 0 \end{aligned}\right\} \text{ active constraints at } x=(0,2).$$

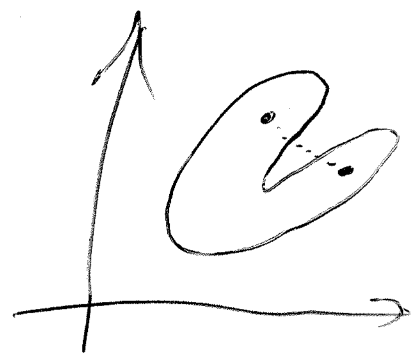
Can have $\Delta x_2 < 0$ and/or $3\Delta x_1 + \Delta x_2 > 0$

At $x = (1,0)$: Need $\Delta x_2 \ge 0$, otherwise don't care.

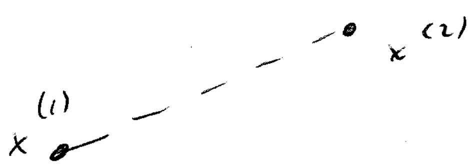
~~[illegible]~~ Convex Sets

A feasible set of an optimization problem is convex if the line segment between any two feasible points is entirely in the feasible region.





Line segment between two points $x^{(1)}$ and $x^{(2)}$:



$$x^{(2)} = x^{(1)} + (x^{(2)} - x^{(1)})$$

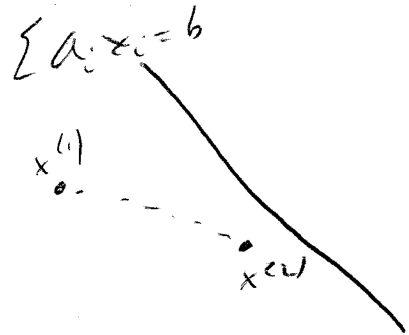
Line segment:

$$x = x^{(1)} + \lambda(x^{(2)} - x^{(1)}), \quad 0 \le \lambda \le 1$$
$$= (1-\lambda)x^{(1)} + \lambda x^{(2)}.$$

If not convex, can get local min that is not global, even for a linear objective function. (Note appears two pages later, so don't need this here.)

Note: If all constraints are $\underline{\text{linear}}$ then the feasible region is convex.

Why?



Consider constraint $\sum_{i=1}^{n} a_i x_i \le b$

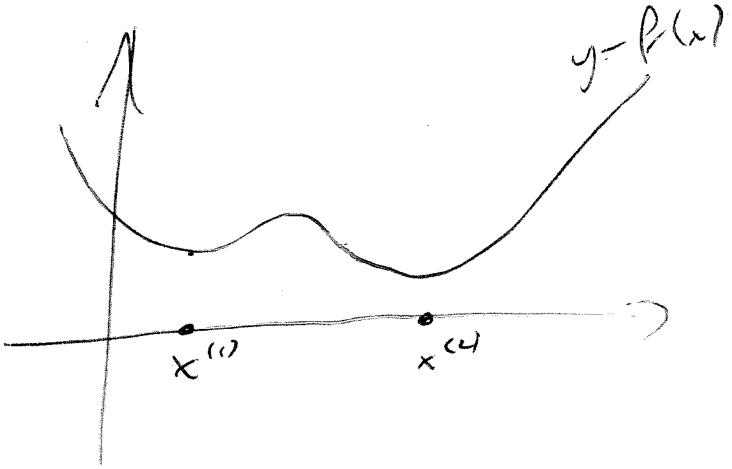
Let $x^{(1)}$ and $x^{(2)}$ satisfy this constraint.

Let $\bar{x} = (1-\lambda)x^{(1)} + \lambda x^{(2)}$, $0 \le \lambda \le 1$, a point on the line segment.

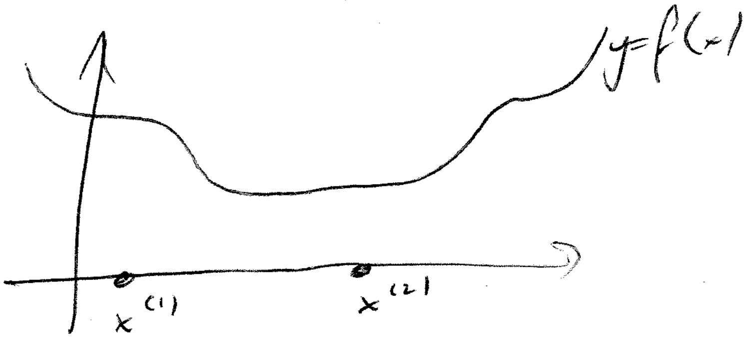
Then
$$\sum_{i=1}^{n} a_i \bar{x}_i = \sum_{i=1}^{n} a_i\left((1-\lambda)x^{(1)} + \lambda x^{(2)}\right)$$
$$= (1-\lambda)\sum_{i=1}^{n} a_i x^{(1)} + \lambda \sum_{i=1}^{n} a_i x^{(2)}$$
$$\le (1-\lambda)\, b + \lambda\, b \qquad \text{since } 0 \le \lambda \le 1.$$
$$= b$$

Unimodal functions

An objective function is <u>unimodal</u> if the straight line from every point in its domain to every better point is an improving direction.



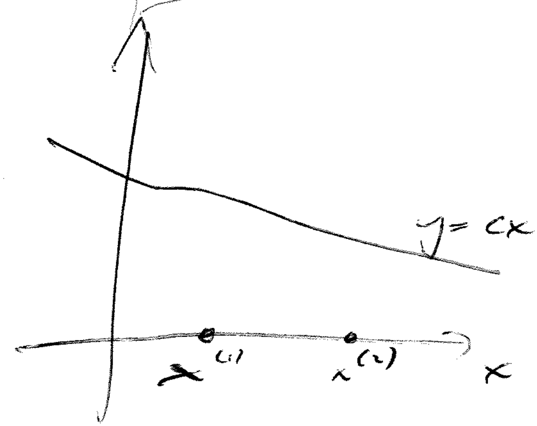
Not unimodal
$x^{(2)}$ is better than $x^{(1)}$, but as move from $x^{(1)}$ to $x^{(2)}$ the function value increases.



Unimodal.

Unimodal: one valley, with sides always going up ~~flat~~ never down.
Note: sides cannot be flat.

Linear objective functions are unimodal



∴ If the objective function of an optimization model is unimodal and the constraints produce a convex feasible set, every local optimum is a global optimum. Such problems can be well solved by improving search. ‖ EXAMPLE: Linear programming.

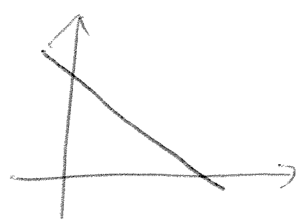
Optimal Solutions

Every local optimum to an LP is a global optimum

A feasible solution to a linear program is a BOUNDARY POINT if at least one inequality constraint that can be strict is satisfied at equality at the given point. Otherwise, it is an INTERIOR POINT.

Eg:

$$x_1 + x_2 \le 10$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$



(5,5) boundary
(5,4) interior

Eg: $x_1 + x_2 \le 10$
$x_1 + x_2 \ge 10$
$x_i \ge 0$.
Boundary points: (10,0), (0,10). Others interior.

Eg:

$$x_1 + x_2 + x_3 \le 10$$
$$x_1 + x_2 \ge 7$$
$$x_3 \ge 3$$
$$x_1 + 2x_2 \le 10$$
$$x_1, x_2, x_3 \ge 0$$

} imply $x_1 + x_2 + x_3 = 10$, $x_1 + x_2 = 7$, $x_3 = 3$.

Interior point: $x_1 = 5,\ x_2 = 2,\ x_3 = 3$

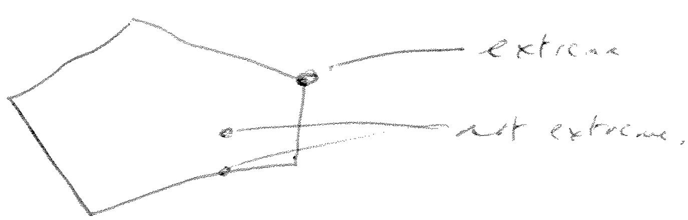
~~$x_1 = 7,\ x_2 = 0,\ x_3 = 3$~~

Boundary point: $x_1 = 4,\ x_2 = 3,\ x_3 = 3$.

or $x_1 = 7,\ x_2 = 0,\ x_3 = 3$.

Defn EXTREME POINTS of convex sets are those that do not lie within the line segment between any two other points in the set.

Eg:



Note: Every optimal solution to a typical linear program will be a boundary point of its feasible region.

Why? ~~Result~~ Push the contour to the boundary.

A "non-typical" LP:

$$\begin{aligned} \min \quad & 0 \\ \text{s.t.} \quad & x_1 + x_2 \le 10 \\ & x_1 \ge 0 \\ & x_2 \ge 0. \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \min \quad & x_1 \\ \text{s.t.} \quad & x_1 \le 5 \\ & x_1 \ge 5 \\ & x_1 \ge 0. \end{aligned}$$

Note: If a linear program has a unique optimal solution, that optimum must ~~be~~ occur at an extreme point of the feasible region.

Why? Again, push the contour to the boundary.

Note: If ~~a~~ a linear program has an optimal solution, it has one at an extreme point.