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Name:

Math Models of Operations Research, MATP 4700/ DSES 4770.

Second Exam, Friday, November 21, 2008.

You may use any result from your notes or a homework that is clearly stated. You may use one sheet of handwritten notes, but no other sources. The exam consists of five questions, and lasts one hundred and ten minutes.

Solutions.

Q1	/ 20
Q2	/ 20
Q3	/ 10
Q4	/ 20
Q5	/ 30
Total	/ 100
Grade	

1. (20 points)

   The Wonder Waste disposal company has 5 truckloads of nuclear waste and 5 truckloads of hazardous chemical wastes that must be moved from its current cleanup site to nuclear and chemical disposal facilities, respectively. The following table shows that many of the available roads are restricted for one or the other type of waste.

Road			
From	To	Nuclear OK	Chemical OK
Site	NDisp	Yes	No
Site	CDisp	Yes	Yes
Site	Inter	No	Yes
NDisp	CDisp	No	Yes
CDisp	Inter	Yes	No
Inter	NDisp	Yes	Yes

   Also Wonder Waste wants to distribute any risk by allowing no more than half the ten total truckloads on any road. One particular road, the link from crossing Inter to the nuclear disposal facility, is especially well suited to hazardous transfer because it runs through very remote areas. Wonder Waste seeks a feasible shipping plan that maximizes use of that road.

   (a) Formulate a linear program to determine an optimal way to carry the flows.

   (b) Show that your LP can be represented as a multicommodity flow problem by sketching the corresponding digraph and labeling with costs, capacities, and net demands.

   (c) Explain why your multicommodity model would give meaningless results if all flow were combined into a single commodity.

2. (20 points. Each part is worth 10 points.)

The linear programming problem

$$\begin{array}{lrcrcrcrcl}
\text{min} & -3x_1 & + & x_2 & + & 4x_3 & + & 7x_4 & & \\
\text{subject to} & 5x_1 & + & 3x_2 & - & x_3 & + & 3x_4 & = & b_1 \\
 & 4x_1 & + & x_2 & + & 3x_3 & - & 2x_4 & = & b_2 \\
 & & & & & & & x_i & \geq & 0 \quad i = 1, \ldots, 4
\end{array}$$

has a nondegenerate optimal solution where x_1 and x_2 are basic variables.

(a) What is the optimal dual solution? (Hint: the dual problem to $\min\{c^T x : Ax = b, x \geq 0\}$ is $\max\{b^T y : A^T y \leq c\}$.)

(b) Show that only one dual slack variable is positive. What do you conclude?

(a) Dual:

$$\begin{array}{lrl}
\max & b_1 y_1 + b_2 y_2 & \\
\text{s.t.} & 5y_1 + 4y_2 & \leq -3 \\
 & 3y_1 + y_2 & \leq 1 \\
 & -y_1 + 3y_2 & \leq 4 \\
 & 3y_1 - 2y_2 & \leq 7
\end{array}$$

Complementary slackness gives

$$5y_1 + 4y_2 = -3 \quad (1)$$
$$3y_1 + y_2 = 1 \quad (2)$$

$$4(2) - (1) \Rightarrow 7y_1 = 7 \Rightarrow \boxed{y_1 = 1 \Rightarrow y_2 = -2}$$

(b) Dual slacks: $4 + y_1 - 3y_2 = 11 > 0$

$7 - 3y_1 + 2y_2 = 0$

So have multiple optimal primal solutions.

3. (10 points)

A transportation problem with three supply nodes and four demand nodes has supplies, demands, and arc costs as indicated below:

		Demand			
		10	30	40	20
Supply	50	8^{10}	3^{30}	1^{10}	10
	20	2	4	4^{20}	7
	30	5	2	6^{10}	5^{20}

Find a basic feasible solution using the Northwest Corner Rule. What are the values of the basic variables in your bfs?

$x_{11} = 10$ $\quad x_{12} = 30$ $\quad x_{13} = 10$

~~$x_{21} = 10$~~ $\quad x_{23} = 20$ $\quad x_{33} = 10$

~~$x_{22} = 10$~~

$x_{34} = 20$.

4. (20 points)

A transportation problem with three supply nodes and four demand nodes has tableau

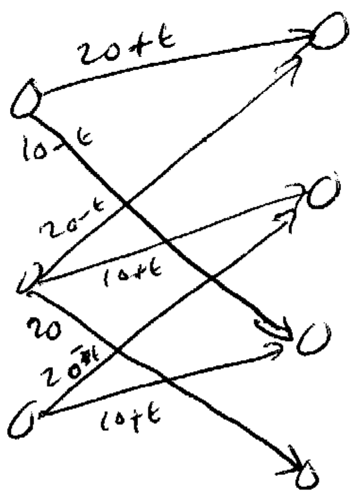
2^{20}	4	4^{10}	7
5^{20}	4^{10}	6	5^{20}
8	3^{20}	1^{20}	4

The large numbers give the costs of the arcs and the superscripts indicate a flow.

(a) (10 points) The given solution is not basic. Find a basic feasible solution that is at least as good as the given solution and that has zero flow on each arc where the given flow is zero.

(b) (10 points) Is the basic feasible solution you found optimal?

(a)



Not bfs: too many arcs.
Adjust flow around cycle.
Cost of cycle:

$2-4+4-3+1-5=-5.$

So increase flow around cycle.

x_{13} becomes zero.

(b)

2^{30}	4	4	7	$u_1=0$
5^{10}	4^{20}	6	5^{20}	$u_2=3$
8	3^{10}	1^{20}	4	$u_3=2$
$v_1=2$	$v_2=1$	$v_3=-1$	$v_4=2$	

Reduced costs:

0	3	5	5
0	0	4	0
4	0	0	0

All reduced costs ≥ 0, so optimal.

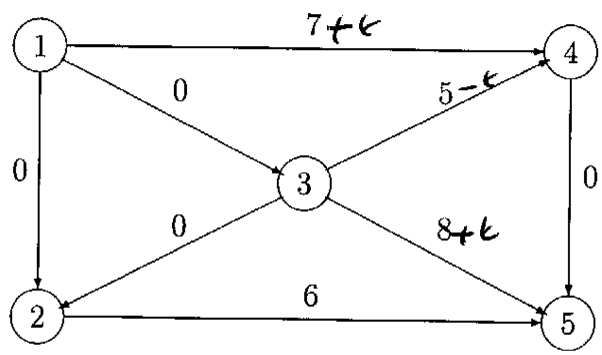
5. (30 points; each part is worth 10 points.)

   Let $G = (V, A)$ be the following directed graph, where the label on each edge gives the flow on that edge. Nodes 1, 2, and 3 are supply nodes, with supplies of 7, 6, and 13, respectively. Nodes 4 and 5 are demand nodes, with demands of 12 and 14, respectively. The objective is the usual one of meeting the demand requirements at minimum cost. There are no capacity restrictions on the arcs. The linear programming formulation can be written compactly as

$$\begin{array}{lllll} \min & \sum_{(i,j)\in A} c_{ij}x_{ij} & & & \\ \text{subject to} & \sum_{j:(i,j)\in A} x_{ij} & - \sum_{j:(j,i)\in A} x_{ij} & = b_i & \forall i \in V \\ & & x_{ij} & \geq 0 & \forall (i,j) \in A \end{array}$$

   where c_{ij} is the cost of shipping one unit along arc (i, j) and b_i is the net supply at node i. Thus, $b_i \geq 0$ for nodes 1, 2, and 3, and negative for the remaining nodes.

   The given flow is optimal, with the basic variables corresponding to arcs $(1, 4)$, $(3, 4)$, $(3, 5)$, and $(2, 5)$. The optimal dual solution is $y_1 = 11$, $y_2 = 4$, $y_3 = 10$, $y_4 = 3$, and $y_5 = 0$.



   (a) How small can the cost c_{32} of arc $(3, 2)$ be if this solution is optimal?

   (b) Assume the supply at node 1 and the demand at node 5 are each increased by one. Assuming the optimal set of basic variables is unchanged, what is the change in the objective function value?

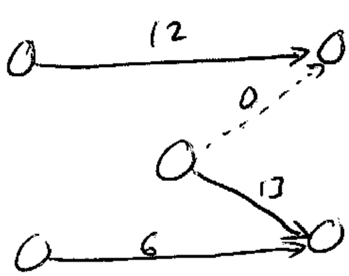
   (c) Let t be the increase in the supply at node 1 and the demand at node 5. What is the maximum possible value of t such that the set of optimal basic variables is unchanged? What is the corresponding optimal solution? Would you expect the rate of change in the optimal value to be larger or smaller for larger values of t?

(a) Need $c_{ij} \geq y_i - y_j$, so $c_{32} \geq y_3 - y_2 = 10 - 4 = 6$

(b) Look at shadow prices. b_1 increases by 1, b_5 decreases by 1. So cost increases by $y_1 - y_5 = 11 - 0 = 11$.

(c) If $t \geq 5$ then $x_{34} \leq 0$. So Need $\boxed{t \leq 5}$

Optimal soln:



Rate of change will be greater, because a worse path must now be found.