A

Name:

Math Models of Operations Research, MATP 4700/ DSES 4770.

First Exam, Tuesday, September 30, 2008.

You may use any result from your notes or a homework that is clearly stated. You may use one sheet of handwritten notes, but no other sources. The exam consists of five questions, and lasts one hundred and ten minutes.

Q1		/ 15
Q2		/ 25
Q3		/ 20
Q4		/ 20
Q5		/ 20
Total		/ 100
Grade		

Solutions.

1. (15 points: each part is worth 5 points.)

The following tableau represents a linear program in standard form:

		x_1	x_2	x_3	x_4	x_5	x_6
	3	0	0	0	0	2	0
12/8	12	8	0	−2	1	6	0
1/1	1	1	1	−1	0	0	0
—	10	−3	0	0	0	10	1

(a) Give an optimal basic feasible solution to this linear program.

(b) Find another optimal basic feasible solution.

(c) Show that the set of optimal solutions is unbounded.

(a) $x_2 = 1, \quad x_4 = 12, \quad x_6 = 10, \qquad x_1 = x_3 = x_5 = 0$

(b) Pivot to have x_1 enter the basis

3	0	0	0	0	2	0
4	0	−4	2	1	6	0
1	1	1	−1	0	0	0
13	0	3	−3	0	10	1

Alternative optimal solution: $x_1 = 1, \; x_4 = 4, \; x_6 = 13, \quad x_2 = x_3 = x_5 = 0$

✓ Feasible in original tableau.

(c) From original tableau, get ray when try to introduce x_3 into basis.

Ray: $d = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$. Can go in this direction arbitrarily far, and remain optimal.

2. (25 points.)

The following tableau represents a linear program in standard form:

	x_1	x_2	x_3	x_4	~~x_5~~	~~x_6~~
3	-3	0	-1	0	~~-2~~	~~0~~
12	8	0	2	1	~~6~~	~~0~~
0	4	1	-1	0	~~0~~	~~0~~
~~10~~	~~-3~~	~~0~~	~~1~~	~~0~~	~~10~~	~~1~~

(a) (10 points) There are ~~three~~ two nonbasic variables. What are the simplex directions corresponding to each of these nonbasic variables?

(b) (10 points) Find the steplengths in the directions found in part 2a. Hence find the new basic feasible solutions that would be obtained by introducing each of these nonbasic variables.

(c) (5 points) Show that the optimal solution can be found in one iteration by choosing the entering variable appropriately.

(a) x_1: $d^{(1)} = \begin{bmatrix} 1 \\ -4 \\ 0 \\ -8 \end{bmatrix}$ x_3: $d^{(3)} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}$ ~~x_5: $d^{(5)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -6 \\ 0 \\ -10 \end{bmatrix}$~~

(b) x_1: min ratio test gives steplength 0 $(\min\{\frac{12}{8}, \frac{0}{4}, \}$)

x_3: min ratio test gives steplength 6 $(\min\{\frac{12}{2}, - \})$

~~x_5: min ratio test gives steplength 1 $(\min\{\frac{12}{6}, -, \})$~~

(c) Introducing x_1: improvement $= 0 \times 3 = 0$

Introducing x_3: improvement $= 6 \times 1 = 6$

~~Introducing x_5: improvement $= 1 \times 2 = 2$~~

So bring x_3 into basis:

9	1	0	0	$\frac{1}{2}$
6	4	0	1	$\frac{1}{2}$
6	8	1	0	$\frac{1}{2}$
~~4~~	~~-11~~	~~0~~	~~0~~	~~$-\frac{1}{2}$~~

Optimal soln:

$x = (0, 6, 6, 0,)$

Value $= -9$.

3. (20 points)

Consider the following tableau for a standard form linear program:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
3	-3	-1	-1	0	-2	0	-4	-6
2	7	13	-2	1	6	0	2	3
1	-3	-6	1	0	10	1	3	-1

Several simplex pivots are taken. The pivot matrix corresponding to this sequence of pivots is:

$$P = \begin{bmatrix} 1 & 6 & 13 \\ 0 & 1 & 2 \\ 0 & 3 & 7 \end{bmatrix}$$

The new basic sequence consists of x_1 and x_3.

(a) (6 points) What is the basic feasible solution obtained as a result of this sequence of pivots? What is its value?

(b) (7 points) Show that the solution you obtained in part 3a is not optimal.

(c) (7 points) How would you next update the pivot matrix?

PM =

28	0	-1	0	6	164	13	47	-1
4	1	1	0					$+1$
13	0	-3	1					2

(a) $x_1 = 4$, $x_3 = 13$, everything else zero. Value $= -28$

(b) x_2 and x_8 have negative reduced costs

(c) If x_2 enters basis: By min ratio, it replaces x_1. $P_{new} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 13 \\ 0 & 1 & 2 \\ 0 & 3 & 7 \end{bmatrix}$

If x_8 enters basis: By min ratio, it replaces x_1. $P_{new} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 13 \\ 0 & 1 & 2 \\ 0 & 3 & 7 \end{bmatrix}$

4. (20 points: each part is worth 5 points)

The variables in the following linear programming problem have upper bounds:

$$
\begin{array}{lrcrcrcrcr}
\text{min} & 5x_1 & - & 3x_2 & & & & & & \\
\text{subject to} & 3x_1 & - & 2x_2 & + & x_3 & & & = & 16 \\
 & 2x_1 & + & x_2 & & & + & x_4 & = & 9 \\
 & & & & & 0 & \le & x_1 & \le & 4 \\
 & & & & & 0 & \le & x_2 & \le & 3 \\
 & & & & & 0 & \le & x_3 & \le & 25 \\
 & & & & & 0 & \le & x_4 & \le & 3
\end{array}
$$

(a) Show that x_3 must be basic in any basic feasible solution. (Hint: Consider the bounds on x_1 and x_2.)

(b) Find all the basic feasible solutions.

(c) Show that the original linear program is equivalent to the following linear program:

$$
\begin{array}{lrcrcrcl}
\text{min} & & & 5x_1 & - & 3x_2 & & \\
\text{subject to} & 6 & \le & 2x_1 & + & x_2 & \le & 9 \\
 & 0 & \le & x_1 & & & \le & 4 \\
 & 0 & \le & & & x_2 & \le & 3
\end{array}
$$

(For the purposes of this question, two linear programs are equivalent if any feasible solution in one can be mapped to a feasible solution in the other with the same objective value.)

(d) Plot the feasible region of the linear program in part 4c. Mark each of the basic feasible solutions you found in part 4b. Find the optimal solution graphically.

(a) $3x_1 - 2x_2 \le 12 - 0 = 12 \Rightarrow x_3 \ge 4$

$3x_1 - 2x_2 \ge 0 - 6 = -6 \Rightarrow x_3 \le 16 + 6 = 22$

So $4 \le x_3 \le 22$, so x_3 cannot be at its bounds.

(b) ~~Six~~ 12 alternatives:

x_1	x_2	x_3	x_4		
B	LB	B	LB	$(4\frac{1}{2}, 0, 2\frac{1}{2}, 0)$	~~BFS~~ violates x_1 bound
B	LB	B	UB	$(3, 0, 7, 3)$	BFS
B	UB	B	LB	$(3, 3, 13, 0)$	BFS
B	UB	B	UB	$(1\frac{1}{2}, 3, 17\frac{1}{2}, 3)$	BFS
LB	B	B	LB	$(0, 9, 34, 0)$	violates x_3 bound
LB	B	B	UB	$(0, 6, 28, 3)$	violates x_3 bound
UB	B	B	LB	$(4, 1, 6, 0)$	BFS
UB	B	B	UB	$(4, -2, 0, 3)$	violates x_2 bound

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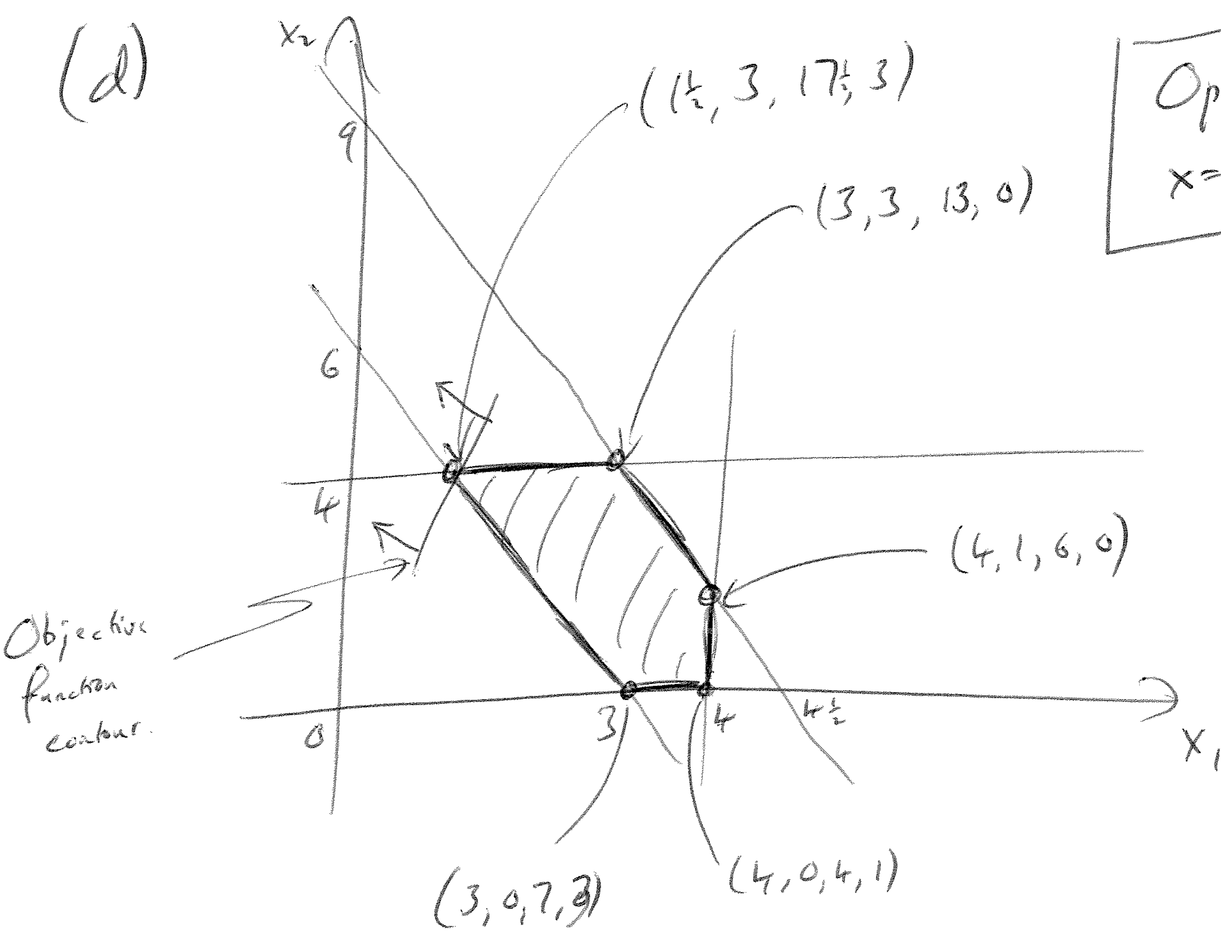
x_1	x_2	x_3	x_4			
LB	LB	B	B	$\Rightarrow$	$(0, 0, 16, 9)$	violates x_4 bound
LB	UB	B	B	$\Rightarrow$	$(0, 3, 22, 6)$	violates x_4 bound
UB	LB	B	B	$\Rightarrow$	$(4, 0, 40, 21)$	BFS ~~violates x_4 bound~~
UB	UB	B	B	$\Rightarrow$	$(4, 3, 10, -2)$	violates x_4 bound

So get 5 basic feasible solutions

(c) First original constraint is redundant, since bounds on $x_1, x_2 \Rightarrow 4 \le x_3 \le 22$.
Then taking x_4 as a slack gives the second formulation.

(d)



Optimal solution:
$x = (1\frac{1}{2}, 3, 17\frac{1}{2}, 3)$

5. (20 points; each part is worth 5 points)

A company manufactures parts $i = 1, \ldots, m$ in weeks $t = 1, \ldots, n$, where each unit of part i requires $a_{i,k}$ units of production resource $k = 1, \ldots, q$ and has value v_i. Production resource capacities b_k cannot be exceeded in any period and part demands $d_{i,t}$ must be met. Express each of the following as linear constraint(s) in these parameters and the nonnegative decision variables $x_{i,t}$:= number of units of i produced in week t and $z_{i,t}$:= inventory of product i held at the end of period t. Initial inventories all = 0.

(a) No production capacity can ever be exceeded.

(b) The total value of held inventories should never exceed 200.

(c) Quantities of each part i available after week 1 should balance with demand and accumulated inventory.

(d) Quantities of each part i available after weeks $2, \ldots, n-1$ should balance with demand and accumulated inventory.

(a) $\sum_i a_{i,k} x_{i,t} \leq b_k$ $\cancel{\forall}$ for $t = 1, \ldots, n$

(b) $\sum_i v_i z_{i,t} \leq 200$ for $t = 1, \ldots, n$

(c) $x_{i,1} = d_{i,1} + z_{i,1}$ for $i = 1, \ldots, m$

(d) $x_{i,t} = d_{i,t} + z_{i,t} - z_{i,t-1}$ for $i = 1, \ldots, m$, $t = 2, \ldots, n-1$.