MATH2800 Introduction to Discrete Structures

First Exam, Tuesday, September 23, 2008.

You may use one sheet of handwritten notes, but no other sources. The exam consists of five questions, and lasts fifty minutes. Please work all five problems. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No calculators are allowed. Please ring your section below:

1: Monday 9am  2: Thursday 9am  3: Monday 2pm  4: Thursday 2pm

<table>
<thead>
<tr>
<th>Q1</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q2</td>
<td>20</td>
</tr>
<tr>
<td>Q3</td>
<td>20</td>
</tr>
<tr>
<td>Q4</td>
<td>20</td>
</tr>
<tr>
<td>Q5</td>
<td>20</td>
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<td>Total</td>
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</tr>
</tbody>
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1. (20 points) A certain island is populated by only two types of people: knights, who always tell the truth, and knaves, who always lie. Two of the inhabitants of the island tell you the following:

A says: Both of us are knights.
B says: A is a knave.

What are A and B? (Note: make sure that you show that your answer is the only consistent solution.)

Suppose A is a knight.

⇒ both A & B are knights by A's statement.
⇒ B statement is true but A is a knave.
⇒ contradiction

∴ A is a knave.
& B is a knight since B's statement is false.

+5 for incorrect guess
+10 for correct unjustified guess

\[\begin{array}{ccc}
A & B & \text{A is a knight.} \\
1 & 1 & \checkmark \\
1 & 0 & \times \\
0 & 1 & \checkmark \\
0 & 0 & \times \\
\end{array}\]
2. (a) (10 points) Convert \((10101001111)_2\) into its hexadecimal expansion.

(b) (10 points) How many zeroes are there at the end of the number \(25^4 \times 48^2\)?

\[
\begin{align*}
25^4 \times 48^2 & = 5^8 \cdot 6^2 \cdot 8^2 = 5^8 \cdot 2^6 \cdot 2^4 \cdot 2^2 \cdot 2^2 \\
& = 5^8 \cdot 2^{10} \cdot 2^2 \\
& = 10^8 \cdot 3^2 \\
& \Rightarrow \text{8 zeroes}.
\end{align*}
\]
3. (20 points) Solve the congruence $4x \equiv 5 \pmod{9}$.

$$\text{gcd}(4, 9) = 1$$

$q = 2(4) + 1$

$$1 = 1(q) - 2(4)$$

$$[1(9) - 2(4)] \equiv 1 \pmod{9}$$

$$-2(4) \equiv 1 \pmod{9}$$

$x \equiv 4(-2) \equiv 1 \pmod{6}$

$x \equiv 4(-2) \equiv 8 \pmod{9}$

$x \equiv -10 \pmod{9}$

$x \equiv 8 \pmod{9}$, for $x = 9k - 10$

-8 for wrong concept of congruence & modulus, but wrong property of x

+8 as $b + km$

-2 for neg. drop,

-2 missing $\pmod{9}$ in answer

-3 wrong $x$

-7 for answer, no verification, saying gcd(a, m) = 1
4. (20 points) Recall that \( A \times B \) denotes the Cartesian product of the two sets \( A \) and \( B \). Let \( A, B, \) and \( C \) be three sets. Prove that
\[
A \times (B \cup C) = (A \times B) \cup (A \times C).
\]

\[\begin{align*}
\text{Proof: Direct} & \quad \forall x \in A \times (B \cup C) \\
& \quad \Rightarrow x = (a, \alpha), \ a \in A, \ \alpha \in B \cup C \\
& \quad \Rightarrow x = (a, b) \text{ or } x = (a, c), \ a \in A, \ b \in B, \ c \in C \\
& \quad \Rightarrow x \in (A \times B) \cup (A \times C) \\
& \quad \therefore A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)
\end{align*}\]
5. (20 points) Prove that the square of any positive odd integer has the form $8m + 1$ for some integer $m$.

Proof (Induction)

Base Case: Let $n = 1$

\[ n^2 = 1 \quad \text{and} \quad 8m + 1 = 8(0) + 1 = 1 \quad \text{for} \quad m = 0 \]

 Induction Hypothesis: Assume $n^2 = 8m + 1$, $n > 0$ odd, $m \in \mathbb{Z}$.

$n = 2k + 1 \Rightarrow n^2 = 4k^2 + 4k + 1$

\[ n + 1 = 2(k+1) + 1 \]

\[ (n+1)^2 = (2(k+1)+1)^2 \]

\[ = (2k+3)^2 \]

\[ = 4k^2 + 12k + 9 \]

\[ = 4k^2 + 4(k+1) + 8k + 8 \]

\[ = 8m + 1 + 8(k+1) + 1 \]

\[ = 8(m+k+1) + 1 \]

\[ = 8(k+1) + 1 \quad \text{for} \quad k+1 \in \mathbb{Z} \]

$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k(k+1)) + 1$

\[ = 8m + 1 \quad \text{for} \quad m = k(k+1) \]

If $k$ odd, then $k+1$ even

If $k$ even, then $k+1$ odd