generalized functions

CHAPTER 8 ~ Relations

§ 8.1 ~ Relations & their Properties

Binary Relation: R s.t. $R \subseteq A \times B$

* $aRb \equiv (a,b) \in R$ and $a \not R b \equiv (a,b) \notin R$

Relation on a set A: R s.t. $R \subseteq A \times A$

- reflexive $\Leftrightarrow \forall a ((a,a) \in R)$
- symmetric $\Leftrightarrow \forall a \forall b ((a,b) \in R \rightarrow (b,a) \in R)$
- antisymmetric $\Leftrightarrow \forall a \forall b ([(a,b) \in R \wedge (b,a) \in R] \rightarrow [a=b])$
- transitive $\Leftrightarrow \forall a \forall b \forall c ([(a,b) \in R \wedge (b,c) \in R] \rightarrow (a,c) \in R)$

Composite of two binary relation R and S, where $R \subseteq A \times B, S \subseteq B \times C$

$\forall a^{\in A} \forall b^{\in B} \forall c^{\in C} ((a,c) \in S \circ R \Leftrightarrow (a,b) \in R \wedge (b,c) \in S)$

§ 8.2 ~ n-ary Relations and Their Applications

n-ary relation: R s.t. $R \subseteq A_1 \times A_2 \times A_3 \times \cdots \times A_n$

- domains $\equiv$ sets $A_1, A_2, \ldots, A_n$ • degree $\equiv$ # of sets/domains

Operations on n-ary relations

- selection operator, S_C ~ pick/keep relations satisfying condition C
- projection operator, $P_{i_1, i_2, \ldots, i_m}$ ~ keeps the domains corresponding to $i_1, i_2, \ldots, i_m$
- join operation, $J_p(R,S)$ ~ merges to relations such that only the complete relations are kept * refer to example 11 (p.534) or #19 (p.536)

§ 8.3 ~ Representing Relations

Matrix Representations

relation $R \Rightarrow M_R = [m_{ij}]$, $m_{ij} = \begin{cases} 1, & (a_i, b_j) \in R \\ 0, & (a_i, b_j) \notin R \end{cases}$ refer to § 8.1

* Note: on paper we stick to binary relations and relations on a set

relation $\bar{R} \Rightarrow M_{\bar{R}} = [m_{ij}]$, $m_{ij} = \begin{cases} 1, & (a_i, b_j) \notin R \\ 0, & (a_i, b_j) \in R \end{cases}$

*recall: $\bar{A} = \{x \mid x \notin A\}$, but here x is an ordered pair and A is a relation

er to Ex 4 p. 539 {* side note: $R_1 \cup R_2 \Rightarrow M_{R_1 \cup R_2} = M_{R_1} + M_{R_2}$, where values > 1 are set to 1

$R_1 \cap R_2 \Rightarrow M_{R_1 \cap R_2} = M_{R_1} \cdot M_{R_2}$, "$\cdot$" denotes entry to entry multiplication

Relations on a set revisited: matrix characteristics of the §8.1 definitions

- reflexive $\Leftrightarrow$ $\text{diag}(M_R) = 1$
- symmetric $\Leftrightarrow$ $M_R = M_R^T$
- antisymmetric $\Leftrightarrow$ $\forall i,j$, where $i \neq j$ either
  ① $m_{ij} = 0$ or $m_{ji} = 0$ OR ② $m_{ij} = m_{ji} = 0$ (i.e. not both 1)

Composite of two binary relations R and S revisited

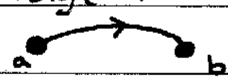
$M_{S \circ R} = M_R \odot M_S$, if an entry value is >1, set it equal to 1

matrix multiplication

* note this works since the set B acts as an intermediate from A to C ($R \subseteq A \times B$, $S \subseteq B \times C$ where $|A| = m$, $|B| = n$, $|C| = p$ so M_R is an $m \times n$ matrix and M_S is an $n \times p$ matrix making $M_{S \circ R}$ an $m \times p$ matrix)

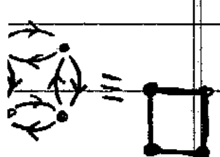
Graphing Representations * obviously directed

- vertex for every element in the set(s)
- edge between vertices if they are a part of the relation s.t. given $(a, b) \in R$, there is a directed edge from initial vertex a to terminal vertex b

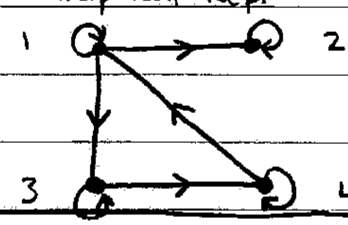


*note: $(a,a) \in R$ is represented by a loop

Relations on a set revisited: graphical characteristics of the §8.1 definitions



- reflexive $\Leftrightarrow$ every vertex of the graph has a loop
- symmetric $\Leftrightarrow$ every pair of vertices has an edge in each direction, which is equivalent to an undirected edge between them.
- anti symmetric $\Leftrightarrow$ arc/edge from i to j not j to i, i.e. a single directed edge between each pair of vertices.

	Matrix Rep.	Graphical Rep.	Properties
Ex: $R = \{(1,1), (1,2), (1,3), (2,2), (3,3), (3,4), (4,1), (4,4)\}$ where $A = \{1,2,3,4\}$	$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ $(1,3) \in R$		* reflexive * anti-symmetric

§8.4 ~ Closures of Relations

Closures
- no such thing as an anti-symmetric closure
- closures add terms, not remove

Let R be a relation on a set (recall : $R \subseteq A \times A$) and P be a property (e.g. reflexive, symmetric, transitive).

If R does not have property P, create a relation S containing the elements of R ($R \subseteq S$) and the missing element required for the relation to have property P.

S will be the "P" closure of the relation R

* reflexive closure: add the missing (a,a) terms $\forall a \in A$

* symmetric closure: add the missing (b,a) terms $\forall (a,b) \in R$

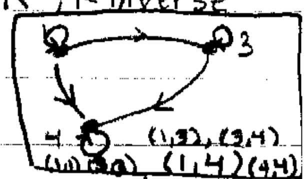
- the set of these missing terms are refered to as R^{-1}, R inverse

* transitive closure: add the missing (x_i, x_n) terms

$\forall (x_i, x_{i+1}) \wedge (x_{i+1}, x_{i+2}) \wedge \dots \wedge (x_{i-n}, x_n) \in R$

* must include (a,a) $\forall a \in A$ (i.e. loops $\forall$ vertices)

* Hint: easiest to find the missing terms if you look at the graphical representation of the relation



adding the red term is transitive closure and defines R^*

*Defn 1 ~ path & circuit/cycle in a directed graph (i.e. traverse vertices following the directed edges)

* number of vertices traverse = path length $= n$

R^n: relation of all paths of length n

*Defn 2 ~ connectivity relation $R^* = \bigcup_{n=1}^{\infty} R^n = \bigcup_{n=1}^{\infty} M_{R^n}$ ← for the matrix representation of relation

(same as transitive closure! :)) ← Thm 2

§8.5 ~ Equivalence Relations

A relation on a set is an equivalence relation if it is reflexive, symmetric & transitive → refer to §8.1 - 8.3

Let R be an equivalence relation on a set. The equivalence class of a, $[a]_R$, is the set of all elements in R related to a

Ex: Congruence modulo 5 (i.e. remainders of real #'s when divided by

#'s having remainder 0 — $[0]_R = \{\dots -10, -5, 0, 5, 10, \dots\}$, $[2]_R = \{\dots -7, -2, 2, 7, 12, \dots\}$

§8.6 ~ Partial Orderings

A relation R on a set A is a partial ordering / partial order if it is reflexive, antisymmetric, & transitive. → refer to § 8.1-8.3

A set 'A' with a partial ordering R is a partially ordered set / poset, (A,R)

* EXAMPLE 1 p.566, $(\mathbb{Z}, \geq)$ is a poset

recall: $(a,b) \in R \equiv aRb$, now we write $a \preceq b$

$a \preceq b$ denotes $(a,b) \in R$ in an arbitrary poset (S,R)

divides symbol

Ex: $(\mathbb{Z}^+, |)$

$(a,b) \in (S, \preceq)$ are comparable if $a \preceq b$ or $b \preceq a$

$a, b \in S$ are incomparable if $a \npreceq b$ & $b \npreceq a$

$(6,12) \in (\mathbb{Z}^+, |)$ $6 | 12$ is comparable

$3,7 \in \mathbb{Z}$ $3 \nmid 7$ and $7 \nmid 3$ is incomparable

* remember relation R, or in this case relation $\preceq$, is a subset of $S \times S$, so there are elements in S that are not related by $\preceq$.

A totally ordered set / linearly order set / chain is a poset $(S, \preceq)$ where every two elements of S are comparable.

* Example (refer to the two above) $(\mathbb{Z}, \geq)$ is totally ordered but not $(\mathbb{Z}^+, |)$

If the poset $(S, \preceq)$ is totally ordered and every non empty subset of S has a least element it is well-ordered

Hasse Diagrams.

Procedure for deriving one:

1. Draw the directed graph for $(S, \preceq)$ → partially ordered set of a relation $\preceq$ on the set S => reflexive, antisymmetric, trans.
   * works best if all edges flow in the same direction (up)
2. Remove loops (they are implied)
3. Remove transitive edges (i.e. if (a,b), (b,c) is there, remove edge (a,c) and if (c,d) is there too, remove (a,d)
4. Eliminate directionality (no arrows)

$\to \forall b \in S\,(a \not\prec b)$

a is maximal in poset $(S, \preceq)$ if $\neg \exists b \in S\,(a \prec b)$ (i.e. a is not related to anything but itself)

" " minimal " " " " $\neg \exists b \in S\,(b \prec a)$ (i.e. nothing is related to a but itself)

$\hookrightarrow \forall b \in S\,(b \not\prec a)$

Ex: ⓐ Determine if the matrix representation of relation $\preceq$ is partially ordered.

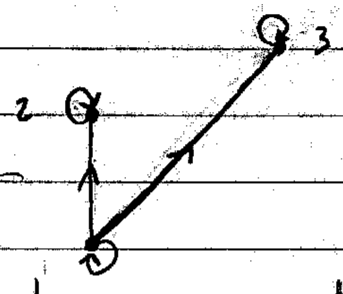
(*refers to #7b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = M_1$ reflective: $diag(M_1) = 1$ ✓

antisymmetric: $\forall i, j$, where $i \neq j$ ✓

① $m_{ij} = 0$ or $m_{ji} = 0$

② $m_{ij} = m_{ji} = 0$

- $(1,2) = 1$, $(2,1) = 0$
- $(1,3) = 1$, $(3,1) = 0$
- $(2,3) = 0$, $(3,2) = 0$



$S = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (1,3), (2,2), (3,3)\}$

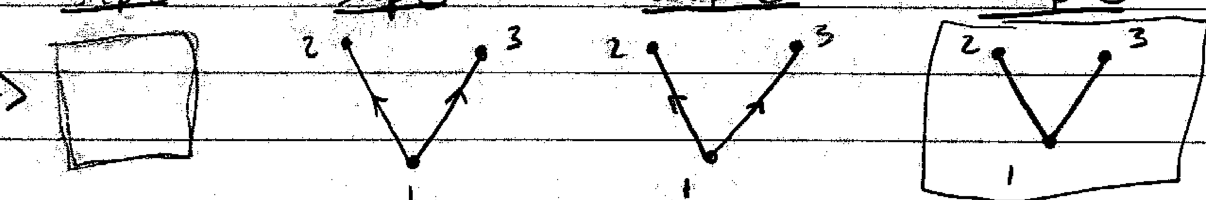
- $\overset{ab}{(1,1)} \wedge \overset{bc}{(1,2)} \wedge \overset{cd}{(2,2)}$ path (a,c)✓, (a,d)✓, (c,d)✓
- $\overset{ab}{(1,1)} \wedge \overset{bc}{(1,3)} \wedge \overset{cd}{(3,3)}$ path (a,c)✓, (a,d)✓, (c,d)✓

transitive ✓

(∴ Relation $\preceq$ on set S is partially ordered, i.e. $(S, \preceq)$)

ⓑ Draw the Hasse Diagram.

step ① step ② step ③ step ④



*remove loops *remove extraneous transitive edges *eliminate directionality

ⓒ Determine the minimal and maximal element(s).

minimal: 1 , maximal: 2, 3