1\. (a) Let $x(\lambda) = x + \lambda d$ for $\lambda \geq 0$. Feasible for all $\lambda \geq 0$.
Let $f(x) = c^T x + \frac{1}{2} x^T Q x$ for all x.

Then
$$f(x(\lambda)) = c^T x + \lambda c^T d + \tfrac{1}{2}(x+\lambda d)^T Q (x+\lambda d)$$
$$= c^T x + \lambda c^T d + \tfrac{1}{2} x^T Q x + \tfrac{1}{2}\lambda^2 d^T Q d + x^T Q d$$
$$= c^T x + \tfrac{1}{2} x^T Q x + \lambda (c + Qx)^T d \quad \text{since } d^T Q d = 0$$
$$\to -\infty \text{ as } \lambda \to \infty \text{ since } (c+Qx)^T d < 0.$$

$\boxed{//}$

(b) Let C be the convex hull of the extreme points of S.
C is a compact set

Any $x \in S$ can be written $x = \bar{x} + \lambda d$,
where $\bar{x} \in C$, $d \in R$, $\lambda \geq 0$, and $\|d\| = 1$.

Since (QP) has unbounded optimal value, we can find points x^i with $f(x^i) \to \infty$.

Write $x^i = \bar{x}^i + \lambda_i d^i$.

Since S and $\{d \in R : \|d\| = 1\}$ are compact, we can find convergent subsequences with $\bar{x}^i \to \hat{x}$, $d^i \to \hat{d}$, $\hat{x} \in C \subseteq S$, $\hat{d} \in R$.

Then $x^i \to \hat{x} + \lambda_i \hat{d}$, so $f(\hat{x} + \lambda_i \hat{d}) \to -\infty$ as $i \to \infty$.

Now, $f(\hat{x} + \lambda_i \hat{d}) = f(\hat{x}) + \lambda_i (c + Q\hat{x})^T \hat{d} + \frac{1}{2} \lambda^2 \hat{d}^T Q \hat{d}$.

For this to be unbounded below, need $\hat{d}^T Q \hat{d} = 0$, $(c + Q\hat{x})^T \hat{d} < 0$. $\boxed{//}$

1 (c) KKT conditions for QP:

$$c + Qx + A^T v = 0 \quad (1)$$
$$v^T(Ax - b) = 0 \quad (2)$$
$$v \geq 0 \quad (3)$$
$$Ax - b \leq 0 \quad (4)$$

Let $\bar{x}$ be a KKT point, let $\bar{d} \in R$.

Then $(c + Q\bar{x})^T \bar{d} = -(A^T v)^T \bar{d}$ from (1)

$$= -v^T A \bar{d}$$

$$\geq 0 \quad \text{since } v \geq 0,\ A\bar{d} \leq 0.$$

(d) $$x = \sum_{i=1}^{k} \lambda_i x^i + d$$

So $$f(x) = \sum_{i=1}^{k} \lambda_i c^T x^i + c^T d + \sum_{i=1}^{k} \lambda_i (Qx^i)^T d + \frac{1}{2} d^T Q d + \frac{1}{2}\left(\sum_{i=1}^{k} \lambda_i x^i\right)^T Q \left(\sum_{i=1}^{k} \lambda_i x^i\right)$$

$$= f\left(\sum_{i=1}^{k} \lambda_i x^i\right) + \left(c + Q\sum_{i=1}^{k} \lambda_i x^i\right)^T d + d^T Q d$$

$$= f\left(\sum_{i=1}^{k} \lambda_i x^i\right) + \sum_{i=1}^{k} \lambda_i (c + Qx^i)^T d + d^T Q d$$

$$\geq f\left(\sum_{i=1}^{k} \lambda_i x^i\right) \quad \text{from (c) and assumptions.}$$

Thus, the optimal solution is a convex combination of the KKT points, and the optimal value is bounded below.

2(a) The feasible region is convex and the origin is a strictly feasible point. Hence the Slater constraint qualification holds.

(b) $L(x, 0) = c^T x$

$$\theta(0) = \inf_{x \in \mathbb{R}^n} c^T x = -\infty \quad : \text{take } x = -\alpha c, \text{ let } \alpha \to +\infty$$

(c) $L(x,u) = c^T x + \frac{1}{2} u_1 x^T M x + \frac{1}{2} u_2 x^T Q x - m u_1 - q u_2$

$$\nabla_x L(x,u) = c + (u_1 M + u_2 Q) x$$

Since $u_1, u_2 \geq 0$ with $(u_1, u_2) \neq (0,0)$, and since Q, M are positive definite, we have $u_1 M + u_2 Q$ is positive definite. Therefore it is invertible.

Thus, the best choice of x is $x(u) = -(u_1 M + u_2 Q)^{-1} c$

$$\begin{aligned} \text{Hence, } \theta(u) &= -c^T(u_1 M + u_2 Q)^{-1} c - m u_1 - q u_2 \\ &\quad + \tfrac{1}{2} u_1 c^T (u_1 M + u_2 Q)^{-1} M (u_1 M + u_2 Q)^{-1} c \\ &\quad + \tfrac{1}{2} u_2 c^T (u_1 M + u_2 Q)^{-1} Q (u_1 M + u_2 Q)^{-1} c \\ &= -c^T (u_1 M + u_2 Q)^{-1} c - m u_1 - q u_2 \\ &\quad + \tfrac{1}{2} c^T (u_1 M + u_2 Q)^{-1} c \\ &= -m u_1 - q u_2 - \tfrac{1}{2} c^T (u_1 M + u_2 Q)^{-1} c \end{aligned}$$

2(d)(i) $u=(1,0)$: $u_1M+u_2Q = M$

so $\theta(u) = -8 - \frac{1}{2}\begin{pmatrix}2 & 2\end{pmatrix}\begin{pmatrix}1 & -2\\ -2 & 5\end{pmatrix}\begin{pmatrix}2\\ 2\end{pmatrix}$

$= -8 - \frac{1}{2}\begin{pmatrix}2 & 2\end{pmatrix}\begin{pmatrix}6\\ -2\end{pmatrix} = -8 - 4$

$= -12$

$\nabla\theta(u) = g(x(u))$

$x(u) = -(u_1M+u_2Q)^{-1}c = -M^{-1}c = -\begin{bmatrix}1 & -2\\ -2 & 5\end{bmatrix}\begin{bmatrix}2\\ 2\end{bmatrix} = \begin{bmatrix}2\\ -6\end{bmatrix}$

$g_1(x(u)) = \frac{1}{2}x^TMx - 8 = \frac{1}{2}\begin{bmatrix}2 & -6\end{bmatrix}\begin{bmatrix}5 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}2\\ -6\end{bmatrix} - 8$

$= \begin{pmatrix}1 & -3\end{pmatrix}\begin{pmatrix}-2\\ -2\end{pmatrix} - 8 = -2+6-8 = -4$

$g_2(x(u)) = \frac{1}{2}x^TQx - 1 = \begin{pmatrix}1 & -3\end{pmatrix}\begin{pmatrix}1 & -2\\ -2 & 5\end{pmatrix}\begin{pmatrix}2\\ -6\end{pmatrix} - 1 = \begin{pmatrix}1 & -3\end{pmatrix}\begin{pmatrix}14\\ -34\end{pmatrix} - 1 = 115$

$u=(0,1)$: $\theta(u) = -1 - \frac{1}{2}\begin{pmatrix}2 & 2\end{pmatrix}\begin{pmatrix}5 & +2\\ +2 & 1\end{pmatrix}\begin{pmatrix}2\\ 2\end{pmatrix} = -1 - \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}14\\ 6\end{pmatrix} = -21$

$x(u) = -Q^{-1}c = -Mc = \begin{pmatrix}-14\\ -6\end{pmatrix}$

$g_1(x(u)) = \frac{1}{2}x^TMx - 8 = \begin{pmatrix}+7 & +3\end{pmatrix}\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}+14\\ +6\end{pmatrix} - 8$

$= \begin{pmatrix}7 & 3\end{pmatrix}\begin{pmatrix}82\\ 34\end{pmatrix} - 8 = 574+102-8 = 668$

$g_2(x(u)) = \frac{1}{2}x^TQx - 1 = \begin{pmatrix}7 & 3\end{pmatrix}\begin{pmatrix}1 & -2\\ -2 & 5\end{pmatrix}\begin{pmatrix}14\\ 6\end{pmatrix} - 1$

$= \begin{pmatrix}7 & 3\end{pmatrix}\begin{pmatrix}2\\ 2\end{pmatrix} - 1 = 19$

$u=(1,1)$: $u_1M+u_2Q = 6I$, so $(u_1M+u_2Q)^{-1} = \frac{1}{6}I$

so $\theta(u) = -\frac{1}{12}c^Tc - 8 - 1 = -\frac{2}{3} - 9 = -9\frac{2}{3}$

$x(u) = -\frac{1}{6}c = -\frac{1}{3}\begin{pmatrix}1\\ 1\end{pmatrix}$.

$g_1(x(u)) = \frac{1}{18}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} - 8 = \frac{10}{18} - 8 = -7\frac{4}{9}$

$g_2(x(u)) = \frac{1}{18}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\ -2 & 5\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} - 1 = \frac{2}{18} - 1 = -\frac{8}{9}$.

2(d)(ii) Thus, $\theta(u) \leq \theta(\bar{u}) + \nabla\theta(\bar{u})^T (u-\bar{u})$

for $\bar{u} = (1,0),\ (0,1),\ (1,1)$

Gives three linear constraints:

$$\theta(u) \leq -12 - 4(u_1 - 1) + 115u_2$$
$$\theta(u) \leq -21 + 668u_1 + 19(u_2 - 1)$$
$$\theta(u) \leq -9\tfrac{2}{3} - 7\tfrac{4}{9}(u_1 - 1) - \tfrac{8}{9}(u_2 - 1)$$

So solve the following LP to get an upper bound:

$$\begin{aligned}
\max\ & \theta \\
\text{s.t. } & \theta \leq -8 - 4u_1 + 115u_2 \\
& \theta \leq -40 + 668u_1 + 19u_2 \\
& \theta \leq -1\tfrac{1}{3} - 7\tfrac{4}{9}u_1 - \tfrac{8}{9}u_2 \\
& u_1 \geq 0 \\
& u_2 \geq 0
\end{aligned}$$

Optimal soln: value -1.7969

$u_1 = 0.0556,\quad u_2 = 0.5587$

Lower bound is best of $\theta(\bar{u})$, namely $-9\tfrac{2}{3}$

3 (a) Take $x=0$: $x^TQx = 0$.

For any other x: $x^TQx \geq 0$.

Hence $x=0$ is optimal with value 0.

(b) (i) KKT conditions:

$$Qx + \alpha x = 0 \;, \quad \alpha(x^Tx - 1) = 0 \;, \quad \alpha \geq 0.$$

So need x to be an eigenvector with eigenvalue $-\alpha$, and $x^Tx = 1$.

Need $\alpha \geq 0$.

So KKT points are exactly the eigenvectors of Q with norm 1 and nonpositive eigenvalue.

(ii) Let λ_1 be the smallest eigenvalue of Q.

Let x' be a corresponding eigenvector of norm 1.

Hessian of Lagrangian is $Q + \alpha I = Q - \lambda_1 I$

This matrix is positive semidefinite, by defn of λ_1.

~~Hence it is psd a~~

It has nullity equal to one, with x' giving a basis for nullspace.

The subspace of interest is $d^Tx' = 0$.

On this subspace we have $d^T(Q-\lambda_1 I)d > 0$, for $d \neq 0$.

Hence the second order sufficient condition is satisfied.

Let $\bar{\lambda}$ be a different eigenvalue, with eigenvector $\bar{x}$.

Hessian of Lagrangian is $Q - \bar{\lambda} I$.

Subspace of interest is $d^T\bar{x} = 0$.

Since evecs are orthogonal, $d = x'$ is valid. Then $x'^T(Q - \bar{\lambda} I)x' = \lambda_1 - \bar{\lambda} < 0$

So: only local mins (and hence global mins) are evecs corresponding to smallest eigenvalue.