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Name:

Nonlinear Programming, MATP6600/DSES6780
Final Exam, Friday, December 7, 2007.

Please do all four problems. You must show all work to obtain full credit. Results from class or the text may be used if properly stated. No books or calculators allowed. The exam lasts one hour and 50 minutes.

Q1		30
Q2		15
Q3		25
Q4		30
Total		100

SOLUTIONS

1. (30 points) Consider the unconstrained problem

$$\min\ \phi(x), \qquad (P)$$

where

$$\phi(x) := \max_{1\le i\le m} f_i(x),$$

and each f_i is C^2 from $\mathbb{R}^n$ to $\mathbb{R}$. Note that in general $\phi(x)$ is not differentiable at points x where the maximum is achieved by two different functions f_i.

(a) (7 points) Assume f_1 is convex and that $\phi(\bar{x}) = f_1(\bar{x})$. Give a valid *linear* inequality for the objective function in terms of x.

(b) (7 points) Express (P) as an equivalent constrained problem (P') involving only smooth functions.

(c) (8 points) If each f_i is convex, what can you say about any local solution to (P') (and hence about any local solution to (P))?

(d) (8 points) Assume constraint qualification holds for the problem (P'). Use the Karush-Kuhn-Tucker conditions to obtain first order necessary conditions for $\bar{x}$ to be a local minimizer of (P).

(a) $\phi(x) \ge f_1(x) \ge f_1(\bar{x}) + \nabla f_1(\bar{x})^T(x-\bar{x})$

(b)
$$\begin{array}{ll} \min & t \\ \text{s.t.} & f_i(x) - t \le 0 \quad i=1,\dots,m \end{array} \qquad (P')$$

(c) (P') is a convex problem, so any local solution to (P') is a global minimizer. Hence, any local solution to (P) is a global minimizer to (P).

(d) $\displaystyle\sum_{i=1}^m \bar{u}_i \nabla f_i(\bar{x}) = 0, \quad 1 - \sum_{i=1}^m \bar{u}_i = 0, \quad \bar{u}_i \ge 0$

$\bar{u}_i(f_i(\bar{x}) - \bar{t}) = 0 \quad \forall i$

$f_i(\bar{x}) - \bar{t} \le 0 \quad \forall i$

2. (15 points; each part is worth 5 points.) Consider the nonlinear programming problem

$$\begin{array}{rrcl} \min & f(x) & := & (2x_1+1)e^{x_2-x_1} \\ \text{s.t.} & x_1 & \geq & 0 \\ & x_2 & \geq & 0 \\ & x & \in & \mathbb{R}^2. \end{array}$$

Note that the constraints are linear, so constraint qualification holds throughout the feasible region. The gradient and Hessian of f are given by

$$\nabla f = e^{x_2-x_1}\begin{bmatrix} 1-2x_1 \\ 2x_1+1 \end{bmatrix} \qquad \nabla^2 f = e^{x_2-x_1}\begin{bmatrix} 2x_1-3 & 1-2x_1 \\ 1-2x_1 & 2x_1+1 \end{bmatrix}.$$

(a) What are the first order necessary Karush-Kuhn-Tucker conditions for this problem?

(b) Show that there are exactly two points which satisfy the first order necessary KKT conditions.

(c) Is $f(x)$ convex in the region $x_1 \geq 2, x_2 \geq 0$?

(a) $$e^{\bar{x}_2-\bar{x}_1}\begin{bmatrix} 1-2\bar{x}_1 \\ 2\bar{x}_1+1 \end{bmatrix} - \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} 1 \\ 2 \end{matrix}$$

$\bar{u}_1\bar{x}_1 = 0, \quad \bar{u}_2\bar{x}_2 = 0, \quad \bar{x}_1 \geq 0, \bar{x}_2 \geq 0, \quad \bar{u}_1 \geq 0, \bar{u}_2 \geq 0$

(b) 2 $\Rightarrow \bar{u}_2 = e^{\bar{x}_2-\bar{x}_1}(2\bar{x}_1+1) > 0$ since $\bar{x}_1 > 0 \Rightarrow \boxed{\bar{x}_2 = 0}$

1 $\Rightarrow \bar{u}_1 = (1-2\bar{x}_1)e^{-\bar{x}_1}$

$\bar{u}_1 > 0 \Rightarrow \bar{x}_1 = 0 \Rightarrow \boxed{\bar{u}_1 = 1}, \boxed{\bar{u}_2 = 1}$ One point: $\boxed{\bar{x}_1 = \bar{x}_2 = 0, \bar{u}_1 = \bar{u}_2 = 1}$

$\bar{x}_1 > 0 \Rightarrow \bar{u}_1 = 0 \Rightarrow \boxed{\bar{x}_1 = \tfrac{1}{2}}$

Second point: $\boxed{\bar{x}_1 = \tfrac{1}{2}, \bar{x}_2 = 0, \bar{u}_1 = 0, \bar{u}_2 = 2e^{-\frac{1}{2}}}$

(c) $$\det(\nabla^2 f) = (e^{x_2-x_1})^2\left[(2x_1-3)(2x_1+1) - (1-2x_1)^2\right]$$

$$= (e^{x_2-x_1})^2\left[4x_1^2 - 4x_1 - 3 - 1 + 4x_1 - 4x_1^2\right]$$

$$= -4(e^{x_2-x_1})^2 < 0. \quad \text{So } \underline{\text{not}} \text{ convex in the region.}$$

3. (25 points; each part is worth 5 points.) Give counterexamples to the following assertions. (Hint: Counterexamples exist in either one or two dimensions for all the assertions.)

   (a) If x satisfies the first order necessary Karush-Kuhn-Tucker conditions for a nonlinear programming problem, then it must be a local minimizer for that problem.

   (b) If x is a local minimizer for a nonlinear programming problem then it must satisfy the first order necessary Karush-Kuhn-Tucker conditions for that problem.

   (c) Two disjoint sets can be separated by a hyperplane.

   (d) The Hessian of a strictly convex function is positive definite everywhere.

   (e) Newton's direction is always a descent direction.

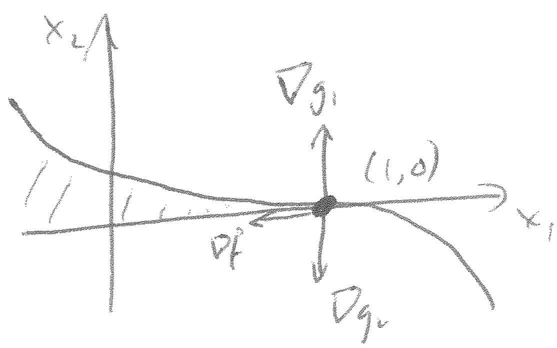
(a) $\min_{x \in \mathbb{R}} \; -x^2$. KKT: $-2x = 0$

Soln: $x = 0$, not local minimizer.

(b) $\min \; -x_1$

s.t. $x_2 - (1-x_1)^3 \leq 0$

$-x_2 \leq 0$



$\bar{x} = (1,0)$ is local minimizer

$0 = \nabla f(\bar{x}) + \bar{u}_1 \nabla g_1(\bar{x}) + \bar{u}_2 \nabla g_2(\bar{x})$

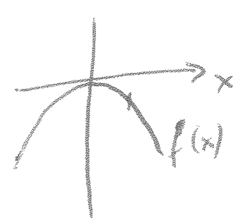
$= \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \bar{u}_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \bar{u}_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$: no solution.

(c) Eg: use nonconvex sets:

(d) Eg: $f(x) = x^4$

$\nabla^2 f(x) = 12x^2$, not positive definite at $x = 0$.

(e) Eg: $f(x) = -x^2$ $\quad \nabla f(x) = -2x$, $\quad \nabla^2 f = -2$



$d = -(\nabla^2 f)^{-1} \nabla f = -\left(\frac{1}{-2}\right)(-2x) = -x$.

Not a descent direction for any x.

4. (30 points.)
Consider the problem

$$\begin{array}{llcll} \min_x & c^T x & & & \\ \text{s.t.} & Ax & = & b & \qquad (CP) \\ & x & \in & K, & \end{array}$$

where x and c are n-vectors, b is an m-vector, A is an $m \times n$ matrix, and K is a convex self-dual cone.

(a) (15 points) A Lagrangian relaxation of (CP) is:

$$\begin{array}{llcll} \min_x & c^T x & + & v^T(b - Ax) & \\ \text{s.t.} & x & \in & K & \qquad (LR(v)) \end{array}$$

where v is an m-vector. Show that the Lagrangian dual problem can be written:

$$\begin{array}{llcllcl} \max_v & b^T v & & & & & \\ \text{s.t.} & A^T v & + & s & = & c & \qquad (LD) \\ & & & s & \in & K & \end{array}$$

(b) (15 points) A barrier formulation for the problem (CP) is

$$\begin{array}{llcll} \min_x & c^T x & - & \mu f(x) & \\ \text{s.t.} & Ax & = & b & \qquad (BCP(\mu)) \\ & x & \in & K & \end{array}$$

where μ is a positive scalar and $f(x) \to -\infty$ as x approaches the boundary of K. Show that the first order KKT optimality conditions of $(BCP(\mu))$ can be expressed as primal feasibility, dual feasibility, plus a complementary slackness condition of the form

$$s = \mu \nabla f(x).$$

(a) $\theta(v) = \min\limits_{x \text{ s.t. } x \in K} c^T x + v^T(b - Ax) = b^T v + \min\limits_{x \in K} (c - A^T v)^T x = \begin{cases} b^T v & \text{if } c - A^T v \in K \\ -\infty & \text{otherwise} \end{cases}$

So Lagrangian dual is:
$$\begin{array}{ll} \max & b^T v \\ \text{s.t.} & c - A^T v - s = 0 \\ & s \in K \end{array}$$

(b) KKT for $x \in \text{int}(K)$:
$c - \mu \nabla f(x) - A^T v = 0$
$Ax = b$

Equivalently: $Ax = b,\ x \in K,$
$c - s - A^T v = 0$
$s = \mu \nabla f(x)$
$s \in K.$

(c) $f(\xi; t) = -\ln(\xi^2 - t^T t)$
$\nabla f(\xi; t) = \frac{-2}{\xi^2 - t^T t} \begin{bmatrix} \xi \\ -t \end{bmatrix}$
$\in \text{int}(K)$ if $(\xi; t) \in \text{int}(K)$
since $(-t)^T(-t) = t^T t < \xi^2$.