ALTERNATING DIRECTION METHOD OF MULTIPLIERS

(from Boyd, Parikh, Chu, Peleato, Eckstein, 2011.)

For problems with equality constraints, with separable objective.

$$\min \; f(x) + g(z) \qquad f, g \text{ convex.}$$
$$\text{s.t. } Ax + Bz = c$$

Use augmented Lagrangian:

$$L_\rho(x, z, y) = f(x) + g(z) + y^T(Ax + Bz - c) + (\rho/2)\|Ax + Bz - c\|_2^2$$

Algorithm updates one set of variables while holding others fixed:

$$x^{k+1} = \operatorname{argmin}_x L_\rho(x, z^k, y^k)$$

$$z^{k+1} = \operatorname{argmin}_z L_\rho(x^{k+1}, z, y^k)$$

$$\cancel{y^{k+1} = \operatorname{argmin}_y L_\rho(x^{k+1}, z^{k+1}, y)}$$

$$y^{k+1} = y^k + \rho(Ax^{k+1} + Bz^{k+1} - c)$$

<u>Convergence</u> If f, g closed, convex, proper, and L_0 has a saddle point

then iterates approach feasibility, objective approaches optimal value.

Eg: LASSO Want sparse approximate solution to system of equations:

$$\min \ \tfrac{1}{2}\|Ax-b\|_2^2 + \lambda\|x\|_1$$

1-norm is good at giving sparse solutions

Equivalently:

$$\min \ \tfrac{1}{2}\|Ax-b\|_2^2 + \lambda\|z\|_1$$

$$\text{s.t.} \quad x - z = 0$$

Exactly in our framework.

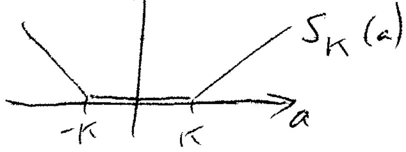
Can solve for x, z, y updates in closed form:

$$x^{k+1} = (A^TA + \rho I)^{-1}(A^Tb + \rho z^k - y^k)$$

$$z^{k+1} = S_{\lambda/\rho}\left(x^{k+1} + y^k/\rho\right)$$

$$y^{k+1} = y^k + \rho\left(x^{k+1} - z^{k+1}\right)$$

where
$$S_K(a) = \begin{cases} a - K & \text{if } a > K \\ 0 & \text{if } |a| \le K \\ a + K & \text{if } a < -K \end{cases} \qquad \text{(soft thresholding)}$$



Fast in practice:
If A is 1500×5000: need just a few seconds.