Conic Optimization

$$\min \sum_{i=1}^{p} c_i^T x_i$$
s.t.
$$\sum_{i=1}^{p} A_i x_i = b$$
$$x_i \in K_i, \quad i=1,\dots,p$$

Where each K_i is a closed convex cone.
x_i is a vector, as is c_i.
A_i is a matrix.

Eg: (i) ~~Eg (i)~~ linear programming:

Can take each x_i to be a <u>scalar</u>, and $K_i = [0, \infty)$.

So then get:
$$\min \sum_{i=1}^{p} c_i x_i$$
s.t.
$$Ax = b$$
$$x \geq 0$$
$$\Big(A = [A_1\ A_2 \dots A_p] \Big)$$
column vector

(ii) Semidefinite programming: (SDP)
Just take one cone K, consisting of all $n \times n$ symmetric psd matrices.

Then C is also an $n\times n$ matrix, and we can write the m linear constraints as:

$$\underbrace{\sum_{i=1}^{n}\sum_{k=1}^{n} A^j_{ik} X_{ik}} = b_j, \quad j=1,\dots,m$$

Frobenius inner product between $n \times n$ matrices A^j and X.
Write as trace $(A^j X)$ or $A^j \circ X$.

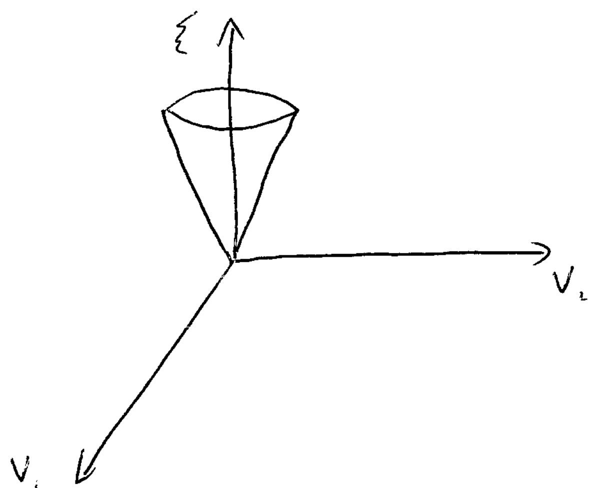
SDP becomes:

$$\begin{aligned} \min \quad & C \bullet X \\ \text{s.t.} \quad & A^j \bullet X = b_j \quad j = 1, \dots, m \\ & X \in K. \end{aligned}$$

(iii) Second order cone programming:

Given a vector $v \in \mathbb{R}^n$ and a scalar ξ, can define a cone as: $\{(\xi; v) : \xi \geq \|v\|_2\}$,

so $\xi^2 \geq \sum_{i=1}^{n} v_i^2$.



"Ice cream cone"
"Lorentz cone"

(iv) Convex quadratic programming:

$$\begin{aligned} \min \ & \tfrac{1}{2} x^T Q x + c^T x \\ \text{s.t.} \ & Ax = b \qquad (QP) \\ & x \geq 0 \end{aligned}$$

Q is psd, so can factor $Q = RR^T$ for some matrix R with linearly independent columns.

So $x^T Q x = x^T R R^T x = \|R^T x\|_2^2$

Thus, (QP) is equivalent to

$$\begin{aligned} \min \ & \tfrac{1}{2} t^2 + c^T x \\ \text{s.t.} \ & Ax = b \\ & v - Rx = 0 \\ & x \geq 0 \\ & (t, v) \in \text{second order cone. (SOC)} \end{aligned}$$

Consider $\tfrac{1}{2} t^2 \leq 2\xi$

Equivalently: $t^2 \leq 4\xi$

Equivalently: $t^2 + \xi^2 - 2\xi + 1 \leq \xi^2 + 2\xi + 1$

Equivalently: $t^2 + (\xi - 1)^2 \leq (\xi + 1)^2$

So QP is equivalent to:

$$\begin{aligned} \min \ & 2\xi + c^T x \\ \text{s.t.} \ & Ax = b, \quad v - Rx = 0, \\ & \lambda = \xi - 1, \quad \mu = \xi + 1 \\ & (t; v) \in SOC, \quad (\mu; t, \lambda) \in SOC. \end{aligned}$$

Self-dual

All these cones are self-dual:

$$K^* = \{z : z^T x \geq 0 \quad \forall x \in K\} = K.$$

~~Barriers~~ Dual problem

So get dual problem:

$$\max \quad b^T y$$
$$\text{s.t.} \quad A_i^T y + s_i = c_i \qquad i = 1, \dots, p$$
$$s_i \in K_i \qquad i = 1, \dots, p.$$

If $\underbrace{\text{Slater's constraint qualification}}$ is satisfied then primal and dual values agree

Have x with $x_i \in \operatorname{int}(K_i)$ $\forall i$,
$\sum_i A_i x_i = b$.

Barriers

All these cones have self-concordant barriers.

a sufficient condition to ensure that the barrier method converges linearly.

LP: $$f(x) = -\sum_i \log(x_i)$$

the eigenvalues of $n \times n$ matrix X

SDP: $$f(X) = -\ln(\det(X)) = -\sum_{i=1}^{n} \ln(\lambda_i(X))$$

SOCP: $$\cancel{f(\xi)}$$
$$f(\xi; v) = -\ln(\xi^2 - v^T v)$$

Then the barrier method converges in polynomial time.