THM (Convergence of Newton's Method)

Assume X_0 is bounded and use directional minimization and f is twice continuously differentiable and its Hessian $\nabla^2 f(x)$ is positive definite at all $x \in X_0$. Then the sequence $\{x_k\}$ generated by Newton's method converges to a minimum x^* of $f(\cdot)$.

Proof

Since use directional minimization, must have decrease, so must have convergence, to a point x^*. Let K be iterates so that $\lim_{\substack{k\to\infty \\ k\in K}} x^k = x^*$

Want to show $\nabla f(x^*) = 0$.

Use contradiction, so assume $\nabla f(x^*) \neq 0$

Let $d^* = -H(x^*)^{-1} \nabla f(x^*) \neq 0$.

From x^*, we'd use directional minimization ~~for~~ along d^* and get step length $\tau^* > 0$, since $d^{*T} \nabla f(x^*) < 0$.

Need to show as $x \to x^*$ get $d \to d^*$ and then get points better than x^*.

Now, as $k \to \infty$, $k \in K$, have $x^k \to x^*$, so $d^k \to d^*$, $H^k \to H^*$

For k sufficiently large, $\|\nabla f(x^k)\| \geq \frac{1}{2} \|\nabla f(x^*)\|$.

Also, X_0 is bounded, so $x^{k+1} \in X^0$. Thus $\tau_k \|\nabla f(x^k)\| \leq \underbrace{\operatorname{diam}(X^0)}_{\text{Largest distance between two points of in } X_0}$

Thus, $0 \leq \tau_k \leq \frac{2}{\|\nabla f(x^*)\|} \operatorname{diam}(X_0)$.

So have a convergent subsequence of τ_k, with limit $\bar{\tau}$

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Thus, $\lim\limits_{\substack{k\to\infty \\ k\in K'}} x^{k+1} = x^* - \bar{\tau}\nabla f(x^*)$

So $\bar{\tau}$ is soln to min $f(x^* + \tau d^*)$, s. $\bar{\tau} > 0$.

Further, $\lim\limits_{\substack{k\to\infty \\ k\in K'}} f(x^{k+1}) < f(x^*)$

So, $\lim\limits_{k\to\infty} f(x^k) \le \lim\limits_{\substack{k\to\infty \\ k\in K'}} f(x^{k+1}) < f(x^*)$ #

$\square$