$\min \; c^T x + \frac{1}{2} x^T B x$ Can be solved by
s.t. $Ax = b$. solving a system of
linear equations:

$\nabla f = c + Bx \qquad \nabla g = A^T$

So:

$$\begin{aligned} c + Bx + A^T v &= 0 \\ Ax &= b \end{aligned}$$

or

$$\begin{aligned} Bx + A^T v &= -c \quad (1) \\ Ax &= b \quad (2) \end{aligned}$$

$(1) \Rightarrow \quad x = -B^{-1}c - B^{-1}A^T v$

Plug in $(2) \Rightarrow: \; -AB^{-1}A^T v - AB^{-1}c = b$

or $\quad v = -(AB^{-1}A^T)^{-1}(b + AB^{-1}c)$.

So

$$x = -B^{-1}c + B^{-1}A^T(AB^{-1}A^T)^{-1}AB^{-1}c + B^{-1}(AB^{-1}A^T)^{-1}b.$$

Lagrange dual of a quadratic program:

$$\min \quad c^T x + \tfrac{1}{2} x^T B x \qquad (CQP)$$
$$Ax \leq b$$
$$x \geq 0$$

$$\theta(u,v) = \min \{ L(x,u,v) \}$$
$$\equiv \min \{ c^T x + \tfrac{1}{2} x^T B x + u^T (Ax - b) - v^T x \}$$

$\cancel{\nabla_x L(x,u,v) =}$

$L(x,u,v)$ is a strictly convex function of x, if B is p.d.

So for given u,v, optimal achieved where $\nabla_x L(x,u,v) = 0$

Now $\nabla_x L(x,u,v) = Bx + c + A^T u - v$

Solution to $\nabla_x L(x,u,v) = 0$ is unique for given u,v

since B square, full rank.

So dual problem is:

$$\max \quad c^T x + \tfrac{1}{2} x^T B x + u^T (Ax - b) - v^T x$$
$$u \geq 0, \; v \geq 0, \; \cancel{Ax \leq b},$$
$$Bx + c + A^T u - v = 0.$$

or $\max \; -\tfrac{1}{2} x^T B x - b^T u$

s.t. $u \geq 0$, $Bx + A^T u \geq -c$ (The [illegible])

$$\min \quad c^T x + \tfrac{1}{2} x^T B x$$

$$Ax \le b$$

$$L(x,u) = c^T x + \tfrac{1}{2} x^T B x + u^T (Ax - b)$$

$$\nabla_x L(x,u) = c + Bx + A^T u \quad \therefore \ \hat{x} = -B^{-1}(c + A^T u)$$

$$\therefore \ \theta(u) = c^T \hat{x} + \tfrac{1}{2} \hat{x}^T B \hat{x} + u^T (A\hat{x} - b)$$

$$= -c^T B^{-1} c - c^T B^{-1} A^T u + c^T B^{-1} A^T u + \tfrac{1}{2} c^T B^{-1} c$$

$$+ \tfrac{1}{2} u^T A B^{-1} A^T u - u^T A B^{-1} c - u^T A B^{-1} A^T u$$

$$- u^T b$$

$$= -\tfrac{1}{2} c^T B^{-1} c - (b^T + c^T B^{-1} A^T) u - \tfrac{1}{2} u^T A B^{-1} A^T u$$

$$\max \quad \theta(u) \quad \text{s.t.} \quad u \ge 0.$$

I.e., another QP.

$$\min \quad c^T x + \frac{1}{2} x^T B x$$ Assume B symmetric.

$$\text{s.t.} \quad Ax \leq b$$
$$x \geq 0.$$

$$L(x, u^1, u^2) = c^T x + \tfrac{1}{2} x^T B x + u^{1T}(Ax - b) - u^{2T} x$$

If B p.d., fine.

~~If B psd~~

O/w: If ~~$\hat{x}$ is an~~ B not psd:

Let $\hat{x}$ ~~satisfy Bx~~ be evec with $-$ve eval.

Then $L(\theta \hat{x}, u^1, u^2) \approx + \frac{1}{2} \theta^2 \hat{\lambda} \to -\infty$ as $\theta \to \infty$.

So $\theta(u) = -\infty$ for all u.

If B psd:

If $x \in N(B)$ then better have $\overbrace{(c^T + u^{1T} A - u^{2T})}^{g(u^1, u^2)} x = 0$, (o/w $\theta(u) = -\infty$)

So can use basis for $N(B)$ to get a finite set of constraints on u.

So $x = \sum \theta_i x^i \quad x^i$: evec

$$L(x, u^1, u^2) = h(\theta, u) = \sum \tfrac{1}{2} \lambda_i \theta_i^2 + \sum \theta_i \, g(u^1, u^2)^T x^i - b^T u^1$$

$$\theta_i = \begin{cases} \text{anything if } g(u^1, u^2)^T x^i = 0 \text{ (and } \lambda_i = 0) \\ -\frac{1}{\lambda_i} g(u^1, u^2)^T x^i & \text{o/w} \end{cases}$$

So $\theta(u) = -b^T u^1 - \frac{1}{2} \sum \frac{g(u^1, u^2)^T x^i}{\lambda_i}$.

$$\min \; \cancel{\tfrac{1}{2}x} \; c^T x + \tfrac{1}{2} x^T B x$$

$$Ax \leq b$$

$$x \geq 0.$$

B sym p.d.

$$\theta(u) = \min_{x \geq 0} \left\{ c^T x + \tfrac{1}{2} x^T B x + u^T (Ax - b) \right\}$$

$$= -b^T u + \min_{x \geq 0} \left\{ (c + A^T u)^T x + \tfrac{1}{2} x^T B x \right\}$$

For a given u, $(c+A^Tu)^T x + \tfrac{1}{2} x^T B x$

~~Dual~~ strictly convex, so achieves its minimum at a point where

$$x \geq 0$$

$$c + A^T u + Bx \geq 0$$

$$(c + A^T u)^T x + x^T B x = 0.$$

$$(c+A^Tu)^T x + x^T B x - \tfrac{1}{2} x^T B x = -\tfrac{1}{2} x^T B x.$$

$$(c + A^T u)^T x + \tfrac{1}{2} x^T B x \leq (c + A^T u)^T x + x^T B x = 0 \quad (\text{Eq only if } x = 0)$$

$$\|$$

$$\cancel{\tfrac{1}{2}} \; \tfrac{1}{2}\left[(c + A^T u)^T x + \tfrac{1}{2} x^T B x \right] + \tfrac{1}{2} (c + A^T u)^T x = \tfrac{1}{2} (c + A^T u)^T x$$

$c^T B^{-1} v - \cancel{c^T B^{-1} v} \quad \cancel{+ c^T B^{-1} v}$

$-c^T B^{-1} A^T u + \cancel{c^T B^{-1} A^T u} \quad \cancel{- c^T B^{-1} A^T u}$

$\cancel{- c^T B^{-1} c} \quad \cancel{+} \frac{1}{2} c^T B^{-1} c$ ✓

$\frac{1}{2} v^T B^{-1} v - v^T B^{-1} v$ ✓

$- v^T B^{-1} A^T u + v^T B^{-1} A^T u + v^T B^{-1} A^T u$ ✓

$\frac{1}{2} u^T A B^{-1} A^T u - u^T A B^{-1} A^T u$ ✓

$\cancel{\frac{1}{2} c^T B^{-1}}$

$- b^T u$ ✓

$$-\frac{1}{2} \begin{pmatrix} u^T & v^T \end{pmatrix} \begin{pmatrix} A \\ I \end{pmatrix} B^{-1} \begin{pmatrix} A^T & I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

$=$ ~~Quadratic term~~

$$\cancel{+} - (b + A B^{-1} c)^T u + c^T B^{-1} v$$

$$- \frac{1}{2} c^T B^{-1} c$$

Find best point on ellipsoid.

$$\min \quad c^T x$$

$$\text{s.t.} \quad x^T M^{-1} x \le 1$$

$$Df = c \qquad Dg = 2M^{-1}x$$

$\therefore \quad c + M^{-1} x u = 0$

$\therefore$ better choose $\hat{x} = \alpha Mc$ for some α

α is chosen so that can take $u > 0$, i.e.,

so that $\hat{x}^T M^{-1} \hat{x} = 1$, and $\alpha < 0$, i.e.

so that $\alpha^2 c^T M c = 1$, i.e. $\alpha = -\sqrt{\frac{1}{c^T M c}}$, so $u = -\frac{1}{2\alpha}$,

$x = -\frac{Mc}{\sqrt{c^T M c}}$.

Dual problem: $L(x,u) = c^T x + u(x^T M^{-1} x - 1)$

$D_x L(x,u) = c + 2uM^{-1}x$. Root at $x = -\frac{1}{2u} Mc$

So $\theta(u) = -\frac{1}{2u} c^T M c + u\left(\frac{1}{4u^2} c^T M c - 1\right) = -\frac{1}{4u} c^T M c - u$,

$\Rightarrow u^* = \frac{\sqrt{c^T M c}}{2}$ ✓.