Quadratic Programming and the Linear Complementarity Problem

(Lemke 65
Cottle & Dantzig 67)

Consider QP

$$\min \quad c^T x + \tfrac{1}{2} x^T B x \qquad (B \text{ symmetric}) \qquad (P)$$
$$Ax \le b \qquad B \;\; n\times n$$
$$x \ge 0 \qquad A \;\; m\times n$$

KKT conditions:

$$\nabla f = c + Bx$$
$$\nabla g = \left(A^T \quad -I \right)$$

from $Ax \le b$ from $-x \le 0$

So KKT conds:

~~$\nabla f + A^T u - I u$~~

~~$\nabla f + A^T u - v =$~~

$$c + Bx + A^T u - v = 0$$
$$u^T (Ax - b) = 0$$
$$v^T x = 0$$
$$u, v \ge 0$$

~~So, if x is an optimal soln to (P) then~~

~~$v + (B \quad A^T$~~

Also, x satisfies

$$Ax + s = b$$

$$x \geq 0 \ , \ s \geq 0$$

slack variables

So, x satisfies:

$$v - \begin{pmatrix} B & A^T \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = c$$

$$s - \begin{pmatrix} -A & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = b$$

$$u^T s = 0, \quad v^T x = 0, \quad u, v, s, x \geq 0$$

ie

$$\begin{pmatrix} v \\ s \end{pmatrix} - \begin{pmatrix} B & A^T \\ -A & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix}$$

$$\begin{pmatrix} v^T & s^T \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = 0$$

$$\begin{pmatrix} v \\ s \end{pmatrix} \geq 0, \quad \begin{pmatrix} x \\ u \end{pmatrix} \geq 0.$$

This is a linear complementarity problem.

Linear complementarity problem

Given $n \times n$ matrix M, $q \in \mathbb{R}^n$

Find $w, z \in \mathbb{R}^n$ s.t.

$$\cancel{w \;\; M}$$

$$w - Mz = q \qquad (1)$$

$$\cancel{w^T \;\; 0}$$

(LCP)

$$w \geq 0, \quad z \geq 0. \qquad (2)$$

$$w^T z = 0 \; (3) \longleftarrow \text{Complementarity condition.}$$

Nonlinear complementarity problem:

Given $f : \mathbb{R}^n \to \mathbb{R}^n$ $\qquad (f(z) = q + Mz)$

find $z \in \mathbb{R}^n$ s.t.

$$f(z) \geq 0$$

$$z \geq 0$$

$$z^T f(z) = 0$$

(w, z) is a feas soln to LCP if it satisfies (1), (2)

Then LCP is feasible. If (3) also holds, (w, z) is a complementary solution.

w_j, z_j are complementary variables; each is complement of other.

(w, z) bfs to LCP if bfs to LP given by ~~satisfies~~ (1), (2) ~~and~~

complementary bfs if also satisfies (3); ie ~~and~~ exactly one of each w_j, z_j is basic.

Return to QP: ([illegible])

Let x be any KKT point (together with u, v, s)

Then
$$f(x) = c^T x + \tfrac{1}{2} x^T B x$$
$$= \tfrac{1}{2} c^T x + \tfrac{1}{2} (Bx + c)^T x$$
$$= \tfrac{1}{2} c^T x + \tfrac{1}{2} (-A^T u + v)^T x \qquad \text{from KKT}$$
$$= \tfrac{1}{2} c^T x + \tfrac{1}{2} v^T x - \tfrac{1}{2} u^T A x$$
$$= \tfrac{1}{2} c^T x + \tfrac{1}{2} v^T x - \tfrac{1}{2} u^T b + \tfrac{1}{2} u^T s$$
$$= \tfrac{1}{2} c^T x - \tfrac{1}{2} u^T b$$

Suggests considering

$$\begin{pmatrix} v \\ s \\ \sigma \end{pmatrix} - \begin{pmatrix} B & A^T & 0 \\ -A & 0 & 0 \\ -c^T & b^T & 0 \end{pmatrix} \begin{pmatrix} x \\ u \\ \tau \end{pmatrix} = \begin{pmatrix} c \\ b \\ 2\delta \end{pmatrix}$$

$$\begin{pmatrix} v \\ s \\ \sigma \end{pmatrix} \geq 0, \quad \begin{pmatrix} x \\ u \\ \tau \end{pmatrix} \geq 0, \quad \begin{pmatrix} v \\ s \\ \sigma \end{pmatrix}^T \begin{pmatrix} x \\ u \\ \tau \end{pmatrix} = 0$$

Complementary solns to this are KKT points with $f(x) \leq \delta$.

So if we can solve LCP efficiently then can find global minimizers for $\ominus$ nonconvex QPs with bdd feas region.

But consider

$$(IP)\quad \begin{array}{ll} \min & c^T x \\ & Ax \le b \\ & x \le e \\ & x \ge 0 \\ & x \text{ integer} \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ll} \min & c^T x + \theta x^T(e-x) \\ & Ax \le b \\ & x \le e \\ & x \ge 0 \end{array}$$

for θ suff large.

Thus to get efficient algs for LCP, treat only subclass of all LCPs (limit class of Ms we consider.)