Example:

Min $\quad x_1^2 + x_2^2$

s.t. $\quad -x_1 - x_2 + 4 \leq 0$

$\quad\quad x_1 + 2x_2 - 8 \leq 0$

Optimum solution: $x_1 = x_2 = 2$. Optimal value $= 8$.

Lagrangian dual problem:

$$\max \quad \theta(u_1, u_2) = \underset{x_1, x_2}{\text{minimum}} \left\{ x_1^2 + x_2^2 + u_1(-x_1 - x_2 + 4) + u_2(x_1 + 2x_2 - 8) \right\}$$

Let $\quad u^1 = (0, 0)$

So $\quad \theta(0,0) = \min_{x_1, x_2} x_1^2 + x_2^2$

Unique soln $x_1 = x_2 = 0$, value 0.

So θ differentiable.

$$\nabla\theta(0) = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$$

From step 2, look in direction $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

$$\theta(4\lambda, 0) = \min_{x_1, x_2} x_1^2 + x_2^2 + 4\lambda(-x_1 - x_2 + 4) \quad (\lambda \text{ fixed})$$

$$= \min_{x_1} (x_1^2 - 4\lambda x_1) + \min_{x_2} (x_2^2 - 4\lambda x_2) + 16\lambda$$

$$= 4\lambda^2 - 8\lambda^2 + 4\lambda^2 - 8\lambda^2 + 16\lambda \quad (\hat{x}_1 = 2\lambda)$$

$$= -8\lambda^2 + 16\lambda$$

Want to maximise this function over λ:

$\lambda_1 = 1$.

$$\therefore \quad u^2 = u^1 + \lambda_1 \begin{pmatrix} \hat{g}_1(x^1) \\ \hat{g}_2(x^1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

$$\theta(4,0) = \min \left(x_1^2 + x_2^2 + 4(-x_1 - x_2 + 4)\right)$$

$$= 8 \quad \text{, achieved at } (2,2)$$

$$g(x^2) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$\therefore \hat{g}(x^2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore \quad u^2$ is optimal.

ε - optimal solutions:

Suppose for a given (u,v) with $u \geq 0$, we have $\hat{x} \in X(u,v)$.

Also, suppose

$$|g_i(\hat{x})| \leq \varepsilon \qquad i \in I$$

$$g_i(\hat{x}) \leq \varepsilon \qquad \forall i$$

$$|h_j(\hat{x})| \leq \varepsilon \qquad \forall j.$$

So $\hat{x}$ is close to feasible (<u>near-feasible</u>).

Let $\bar{x}$ be optimal to primal.

Then

$$f(\hat{x}) + u^T g(\hat{x}) + v^T h(\hat{x}) \leq f(\bar{x}) + u^T g(\bar{x}) + v^T h(\bar{x}) \leq f(\bar{x})$$

$$\therefore \quad f(\hat{x}) \leq f(\bar{x}) + \varepsilon\left[\sum u_i + \sum |v_j|\right].$$

So if $\varepsilon\left[\sum u_i + \sum |v_j|\right]$ small, then $\hat{x}$ is <u>near-optimal</u>.