Properties of the dual function

θ is concave

Theorem $X \subseteq \mathbb{R}^n$, $X \neq \phi$. $f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}^m$, $h: \mathbb{R}^n \to \mathbb{R}^p$,

f, g, h continuous. $\theta(u,v) = \inf\{f(x) + u^T g(x) + v^T h(x) : x \in X\}$

Then θ is a concave function of u and v.

Proof Let $u_1, u_2 \in \mathbb{R}^m_+$, $v_1, v_2 \in \mathbb{R}^p$, $0 < \lambda < 1$.

Then $\theta(\lambda u_1 + (1-\lambda)u_2, \lambda v_1 + (1-\lambda)v_2)$

$$= \inf\{f(x) + \lambda u_1^T g(x) + (1-\lambda)u_2^T g(x) + \lambda v_1^T h(x) + (1-\lambda) v_2^T h(x) : x \in X\}$$

$$= \inf\{\lambda[f(x) + u_1^T g(x) + v_1^T h(x)] + (1-\lambda)[f(x) + u_2^T g(x) + v_2^T h(x)] : x \in X\}$$

$$\geq \lambda \inf\{f(x) + u_1^T g(x) + v_1^T h(x) : x \in X\} + (1-\lambda)\inf\{f(x) + u_2^T g(x) + v_2^T h(x) : x \in X\}$$

$$= \lambda\theta(u_1, v_1) + (1-\lambda)\theta(u_2, v_2)$$

//

From now on, assume X compact (justify this).

Differentiability of θ:

Define $X(u,v) = \{y \in X : y$ minimizes $f(x) + u^T g(x) + v^T h(x)$ over $x \in X\}$

Theorem $X \subseteq \mathbb{R}^n$, $X \neq \phi$, X compact. $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^l$. f, g, h continuous. Suppose $X(\bar{u}, \bar{v})$ is the singleton $\{\bar{x}\}$. Then θ is differentiable at $\bar{u}, \bar{v}$ with gradient ~~$\nabla\theta(w) =$~~

$$\nabla \theta(\bar{u}, \bar{v}) = \begin{pmatrix} g(\bar{x}) \\ h(\bar{x}) \end{pmatrix}.$$

(Proof in book)

Subgradients of θ:

<u>Defn</u> Let ~~ξ~~ $f: \mathbb{R}^n \to \mathbb{R}$ be a concave function. ~~and~~ Then ξ is a subgradient of f at $\bar{x}$ if

$$f(x) \le f(\bar{x}) + \xi^T(x-\bar{x}) \qquad \forall x \in \mathbb{R}^n.$$

Replace f by θ.

<u>Theorem</u> ~~Let~~ $X \subseteq \mathbb{R}^n$, $X \neq \emptyset$, compact.

$f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}^m$, $h: \mathbb{R}^n \to \mathbb{R}^p$ continuous.

Then $X(\bar{u}, \bar{v}) \neq \emptyset$ for any $\bar{u}, \bar{v}$.

In addition, if $\bar{x} \in X(\bar{u}, \bar{v})$, then $(g(\bar{x}), h(\bar{x}))$ is a subgradient of θ at $(\bar{u}, \bar{v})$.

(Proof in book). $\longrightarrow$ See page 133a.

<u>Theorem</u> $X \subseteq \mathbb{R}^n$, $X \neq \emptyset$, compact

$f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}^m$, $h: \mathbb{R}^n \to \mathbb{R}^p$ continuous.

Then ξ is a subgradient of θ at $\bar{u}, \bar{v}$ if and only if ξ belongs to convex hull of ~~$\{\xi(y) : y \in X$~~

$$\{(g(y), h(y)) : y \in X(\bar{u}, \bar{v})\}.$$

(Proof in book).

If $\bar{x} \in X(\bar{u}, \bar{v})$:

$$\theta(u,v) \le f(\bar{x}) + u^T g(\bar{x}) + v^T h(\bar{x})$$ by defn of $\theta(u,v)$

$$= \theta(\bar{u},\bar{v}) + (u-\bar{u})^T g(\bar{x}) + (v-\bar{v})^T h(\bar{x}).$$ by defn of $\theta(\bar{u},\bar{v})$.

$X(\bar{u},\bar{v}) \neq \phi$ by Weierstrass's Thm.

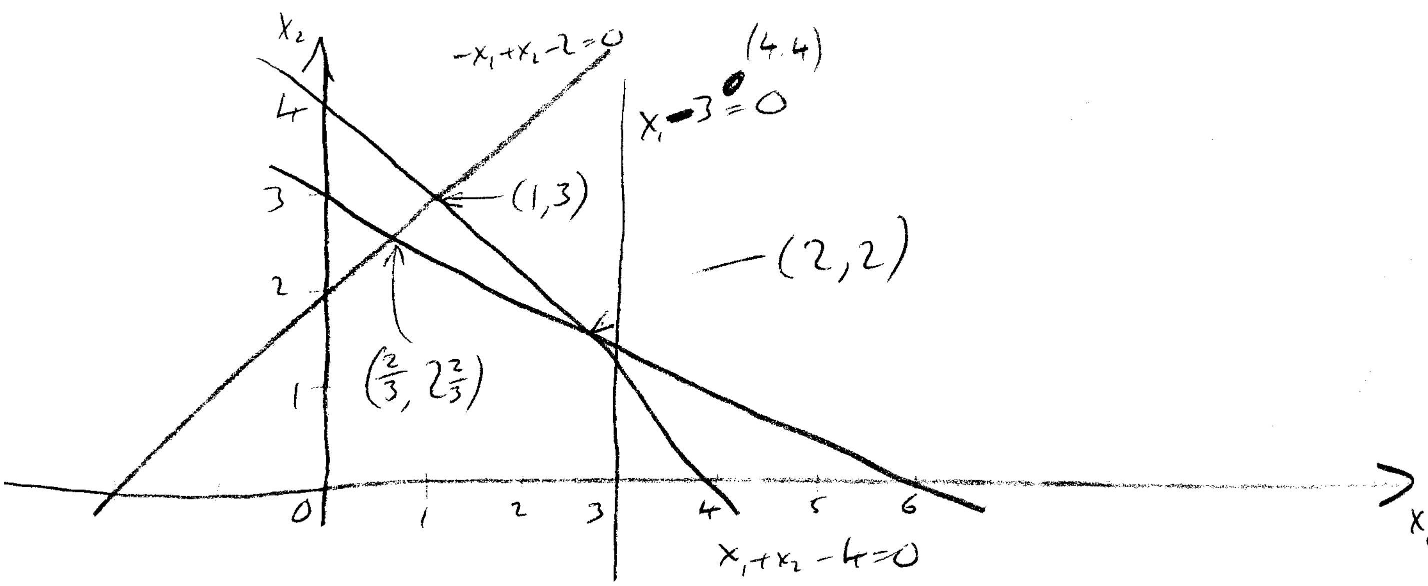
Example

Primal problem:

$$\text{minimize} \quad -(x_1-4)^2-(x_2-4)^2$$

$$\text{s.t.} \quad \begin{aligned} x_1 - 3 &\le 0 && g_1 \\ -x_1+x_2-2 &\le 0 && g_2 \\ -x_1-2x_2+6 &\le 0 && g_3 \end{aligned}$$

$$\left.\begin{aligned} x_1+x_2-4 &\le 0 \\ x_1\ge 0,\ x_2 &\ge 0. \end{aligned}\right\} X.$$



Dual problem:

$$\theta(u_1,u_2,u_3) = \cancel{\inf\{-(4-x_1)^2}$$

$$= \inf\{-(x_1-4)^2-(x_2-4)^2+u_1(x_1-3)+u_2(-x_1+x_2-2)+u_3(-x_1-2x_2+6)$$

$$x_1\ge 0,\ x_2\ge 0,\ x_1+x_2-4\le 0$$

$\bar{u}_1 = 1, \bar{u}_2 = 5, \bar{u}_3 = 0$:

~~Dual problem~~
To evaluate $\theta(\bar{u})$, and find $X(\bar{u})$, need to solve: subproblem.

$$\min \quad -(x_1-4)^2 - (x_2-4)^2 + (x_1-3) + 5(-x_1+x_2-2)$$
$$x_1 + x_2 - 4 \leq 0$$
$$x_1, x_2 \geq 0.$$
(SP)

Obj fn is concave, so min is achieved at extreme point of feasible region.

$x_1 = 0,\ x_2 = 0$: obj fn is $-16-16-3-10 = -45$

$x_1 = 4,\ x_2 = 0$: obj fn is $-16+1-30 = -45$

$x_1 = 0,\ x_2 = 4$: obj fn is $-16 \quad -3+10 = -9$

So $(0,0)$ and $(4,0)$ are both optimal to the <u>subproblem</u>.

$g(0,0) = (-3,-2,6) \qquad g(4,0) = (1,-6,2)$

Thus, $\theta(u_1,u_2,u_3) \leq -45 + u^T \begin{bmatrix} -3 \\ -2 \\ -6 \end{bmatrix}$

and $\theta(u) \leq -45 + u^T \begin{bmatrix} -1 \\ -4 \\ 4 \end{bmatrix}$.

Max $-(x_1-4)^2-(x_2-4)^2$

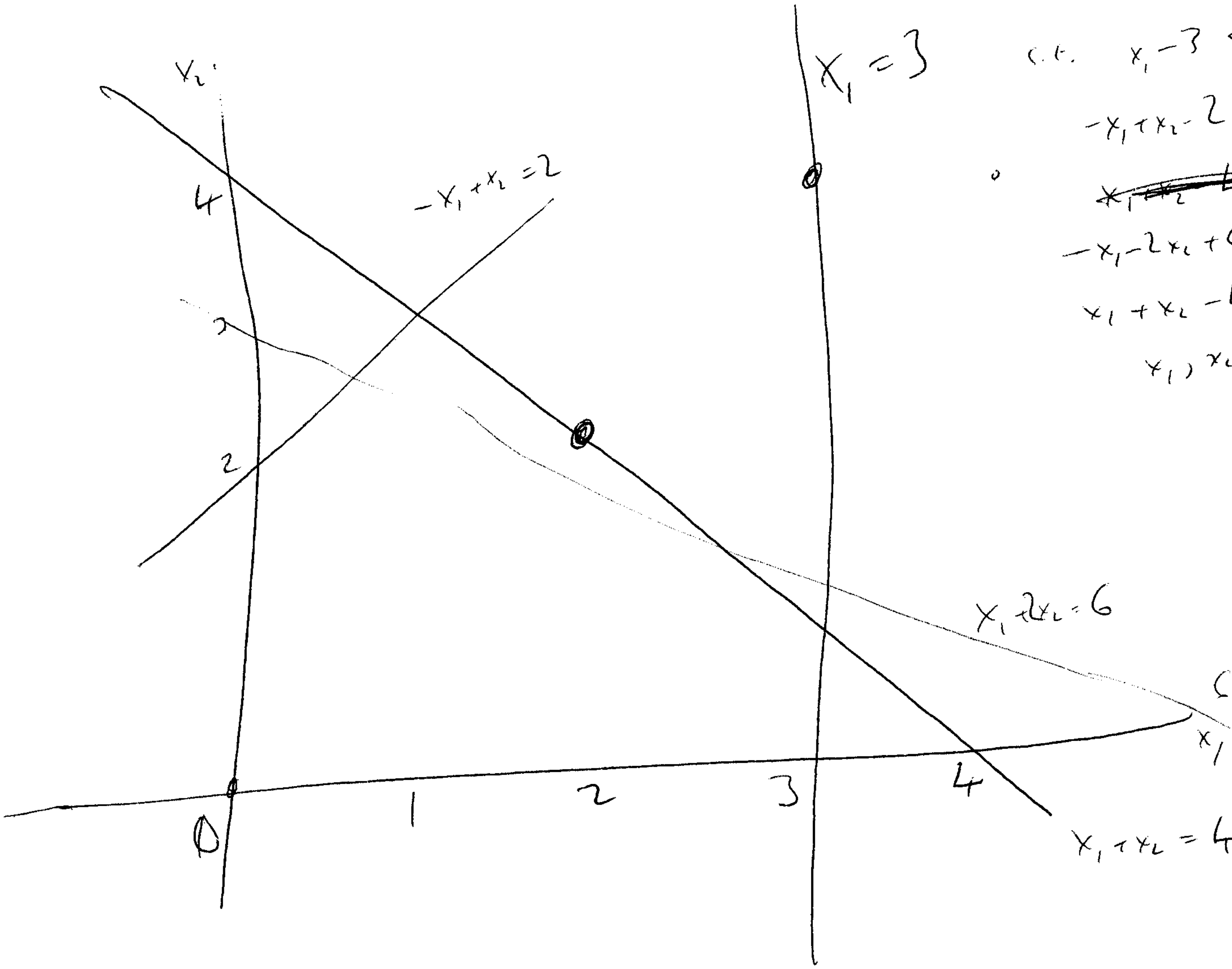
s.t. $x_1-3 \le 0$ g_1

$-x_1+x_2-2 \le 0$ g_2

$\cancel{x_1+x_2-4 \le 0}$

$-x_1-2x_2+6 \le 0$ g_3

$\left.\begin{array}{l} x_1+x_2-4 \le 0 \\ x_1, x_2 \ge 0 \end{array}\right\} X$



$$\theta(u_1,u_2,u_3) = \inf\left\{-(x_1-4)^2-(x_2-4)^2+u_1(x_1-3)+u_2(x_1+x_2-4)+u_3(-x_1-2x_2+6) : x \in X\right\}$$

$\cancel{\theta(u_1,u_2) =}$

β $u_1=1, u_2=5, u_3=0$

$$-x_1 + x_2 = 2$$

$$\underline{x_1 + 2x_2 = 6}$$

$$3x_2 = 8 \qquad \boxed{x_2 = 2\tfrac{2}{3}, \quad x_1 = \tfrac{2}{3}}$$

$$\nabla f = \begin{pmatrix} -2\,(x_1 - 4) \\ -2\,(x_2 - 4) \end{pmatrix}$$

$$\nabla g_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \nabla g_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \nabla g_3 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

So want soln to

$$-2\left(\tfrac{2}{3} - 4\right) - u_2 - u_3 = 0$$

$$-2\left(2\tfrac{2}{3} - 4\right) + u_2 - 2u_3 = 0$$

i.e.
$$u_2 + u_3 = 2\left(4 - \tfrac{2}{3}\right) = \tfrac{20}{3}$$

$$-u_2 + 2u_3 = 2\left(4 - 2\tfrac{2}{3}\right) = \tfrac{8}{3}$$

$$\therefore\ 3u_3 = \tfrac{28}{3} \quad \therefore\ u_3 = \tfrac{28}{9}$$

Also
$$-3u_2 = \tfrac{-32}{3} \quad \therefore\ u_2 = \tfrac{32}{9}$$

$$-32 + 56 = 24 = 8.3$$

Lagrangian dual of LP

$$\min \; c^T x \qquad\qquad \min \; c^T x$$
$$Ax = b \quad \longleftrightarrow \quad b - Ax = 0$$
$$x \geq 0. \qquad\qquad -x \leq 0$$

Alternatively: take $X = \mathbb{R}^n_+$, rather than $\mathbb{R}^n$.

$$L(x,u,v) = c^T x + v^T(b - Ax) - u^T x$$

$$\theta(u,v) = \inf_{x \in X} \{L(x,u,v)\}$$

$$= \inf_x \{b^T v + c^T x - (A^T v)^T x - u^T x\}$$

$$= \inf_x \{b^T v + (c - A^T v - u)^T x\}$$

$$= \begin{cases} b^T v & \text{if } c - A^T v - u \geq 0 \\ -\infty & \text{[redacted] otherwise} \end{cases}$$

$$\max_{u \geq 0} \theta(u,v) \quad = \quad \max_{c - A^T v - u \geq 0,\; u \geq 0} b^T v$$

$$= \max_{A^T v \leq c} b^T v$$