Saddle Point Theorem:

$X \subseteq \mathbb{R}^n$, $X \neq \emptyset$

$f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}^m$, $h: \mathbb{R}^n \to \mathbb{R}^p$.

Suppose $\exists\, \bar{x} \in X$, $\bar{u}, \bar{v}$ with $\bar{u} \geq 0$ such that

$$(*) \qquad L(\bar{x}, u, v) \leq L(\bar{x}, \bar{u}, \bar{v}) \leq L(x, \bar{u}, \bar{v}) \quad \forall x \in X,$$
$$\forall u, v \text{ with } u \geq 0.$$

Then $\bar{x}$ and $(\bar{u}, \bar{v})$ solve the primal and dual problems respectively, and $f(\bar{x}) = \theta(\bar{u}, \bar{v})$.

Conversely, suppose X, f, g convex, h affine, $0 \in \operatorname{int} h(X)$,

$\exists\, \hat{x}$ with $g(\hat{x}) < 0$ and $h(\hat{x}) = 0$.

If $\bar{x}$ is optimal for the primal problem, then $\exists\, (\bar{u}, \bar{v})$ with $\bar{u} \geq 0$ such that $(*)$ holds.

Proof

$\Rightarrow$: Assume $\exists\ \bar{x}, \bar{u}, \bar{v}$ satisfying saddle point condition. Get $\theta(\bar{u},\bar{v}) = L(\bar{x},\bar{u},\bar{v})$ immediately

Feasibility of $\bar{x}$:

Then $f(\bar{x}) + u^T g(\bar{x}) + v^T h(\bar{x}) = L(\bar{x}, u, v) \leq L(\bar{x}, \bar{u}, \bar{v})$

True $\forall u \geq 0$ $\therefore$ Must have $g(\bar{x}) \leq 0$

True $\forall v$ $\therefore$ must have $h(\bar{x}) = 0$.

$\therefore$ $\bar{x}$ primal feasible. ~~$\Rightarrow$ [illegible] $f(\bar{x}) \leq L(\bar{x}, u, v)$~~

Optimality:

Also, true for $u = 0$, ie

$$f(\bar{x}) + \bar{v}^T h(\bar{x}) \leq f(\bar{x}) + \bar{u}^T g(\bar{x}) + \bar{v}^T h(\bar{x})$$

$\therefore$ $\bar{u}^T g(\bar{x}) \geq 0$.

Since $\bar{u} \geq 0$, $g(\bar{x}) \leq 0$, must have $\bar{u}^T g(\bar{x}) = 0$

Thus

$$\begin{aligned} f(\bar{x}) &= f(\bar{x}) + \bar{u}^T g(\bar{x}) + \bar{v}^T h(\bar{x}) \\ &= L(\bar{x}, \bar{u}, \bar{v}) \\ &\leq L(x, \bar{u}, \bar{v}) \quad \forall x \in X \end{aligned}$$

~~$f(\bar{x}) + \bar{u}^T g(\bar{x}) + \bar{v}^T h(\bar{x})$~~

~~$f(\bar{x})$ [illegible] all feasible x~~

skip to here.

~~True $\forall x \in X$~~

$\therefore$ $\theta(\bar{u}, \bar{v}) = \inf\{L(x, \bar{u}, \bar{v}) : x \in X\} \geq f(\bar{x})$.

$\therefore$ by weak duality,

$\bar{x}$ is primal optimal, $\bar{u}, \bar{v}$ is dual optimal. //

$\Longleftarrow$: $\bar{x}$ primal optimal

By strong duality theorem, $\exists\ \bar{u}, \bar{v}$ s.t.

$$f(\bar{x}) = \theta(\bar{u},\bar{v}) \cancel{= \inf\{L(x,\bar{u},\bar{v}) : x \in X\}}$$

and $\bar{u}^T g(\bar{x}) = 0$.

$$f(\bar{x}) = \theta(\bar{u},\bar{v}) \leq L(x,\bar{u},\bar{v}) \quad \forall x \in X$$

$$\cancel{f(x)} \cancel{= f(x)}$$

But also,
$$f(\bar{x}) = f(\bar{x}) + \underbrace{\bar{u}^T g(\bar{x})}_{=0} + \underbrace{\bar{v}^T h(\bar{x})}_{=0}$$
$$= L(\bar{x},\bar{u},\bar{v})$$

ie $L(\bar{x},\bar{u},\bar{v}) \leq L(x,\bar{u},\bar{v}) \quad \forall x \in X$.

Also,
$$L(\bar{x},u,v) = f(\bar{x}) + \underbrace{u^T}_{\geq 0} \underbrace{g(\bar{x})}_{\leq 0} + v^T \underbrace{h(\bar{x})}_{=0}$$
$$\leq f(\bar{x})$$
$$= L(\bar{x},\bar{u},\bar{v})$$

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Relationship between Saddlepoint criteria and the KKT conditions:

Theorem Let $S = \{x \in X : g(x) \le 0,\ h(x) = 0\}$

Suppose $\bar{x}$ satisfies KKT condns for primal problem.

~~f, g_i~~ $I = \{i : g_i(\bar{x}) = 0\}$

f, g_i convex $\forall i \in I$

If $\bar{v}_j \neq 0$ then h_j affine.

Then $(\bar{x}, \bar{u}, \bar{v})$ satisfy saddlepoint conditions

$$L(\bar{x}, u, v) \le L(\bar{x}, \bar{u}, \bar{v}) \le L(x, \bar{u}, \bar{v}) \qquad \forall x \in X, \ \forall u, v \text{ with } u \ge 0$$

Conversely: Suppose $(\bar{x}, \bar{u}, \bar{v})$ with $\bar{x} \in \operatorname{int} X$, $\bar{u} \ge 0$ satisfy saddlepoint conditions. Then $\bar{x}$ is feasible to primal problem, and $\bar{x}, \bar{u}, \bar{v}$ satisfy KKT conditions.

<u>Proof</u>

$\Rightarrow$: $\bar{x}, \bar{u}, \bar{v}$ satisfy KKT conditions

Then
$$\left.\begin{array}{ll} f(x) \geq f(\bar{x}) + \nabla f(\bar{x})^T(x-\bar{x}) & \\ g_i(x) \geq g_i(\bar{x}) + \nabla g_i(\bar{x})^T(x-\bar{x}) & \text{for } i \in I \\ h_j(x) = h_j(\bar{x}) + \nabla h_j(\bar{x})^T(x-\bar{x}) & \text{if } v_j \neq 0 \end{array}\right\} \forall x \in X$$

Multiply by $\bar{u}, \bar{v}$, add:

$$L(x,\bar{u},\bar{v}) = f(x) + \bar{u}^T g(x) + \bar{v}^T h(x) \geq f(\bar{x}) + \bar{u}^T g(\bar{x}) + \bar{v}^T h(\bar{x}) + (\nabla f(\bar{x}) + \nabla g(\bar{x})^T \bar{u} + \nabla h(\bar{x})^T \bar{v})^T (x-\bar{x})$$

$$= L(\bar{x},\bar{u},\bar{v})$$

$L(\bar{x},u,v) \leq L(\bar{x},\bar{u},\bar{v})$ follows by same argument as previous theorem
(ie, since $h(\bar{x}) = 0$, $g(\bar{x}) \leq 0$, $u \geq 0$)

$\Leftarrow$: $\bar{x}, \bar{u}, \bar{v}$ satisfy saddle point condition.

Get feasibility, complementary slackness as for previous theorem.

$L(\bar{x},\bar{u},\bar{v}) \leq L(x,\bar{u},\bar{v})$

$\therefore$ $\bar{x}$ solves the problem : $\min L(x,\bar{u},\bar{v}) : x \in X$

Now $\bar{x} \in \operatorname{int} X$ $\therefore$ $\nabla_x L(\bar{x},\bar{u},\bar{v}) = 0$

ie $\nabla f(\bar{x}) + \nabla g(\bar{x})^T \bar{u} + \nabla h(\bar{x}) \bar{v} = 0.$

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Provides some motivation for choice of dual problem:

We know an optimal solution will satisfy KKT conditions

So must have $\nabla_x L(\bar{x}, \bar{u}, \bar{v}) = 0$,

so $\bar{x}$ is a minimizer of $L(x, u, v)$ for some u, v.

(under convexity assumptions)

Also, $\nabla_u L(\bar{x}, \bar{u}, \bar{v}) = g(\bar{x})$

$\nabla_v L(\bar{x}, \bar{u}, \bar{v}) = h(\bar{x})$

Since $L(\bar{x}, \bar{u}, \bar{v})$ maximizes $L(\bar{x}, u, v)$,

must have $\nabla_v L(\bar{x}, \bar{u}, \bar{v}) = 0$

and $\nabla_{u_i} L(\bar{x}, \bar{u}, \bar{v}) \begin{cases} = 0 & \text{if } u_i > 0 \\ \leq 0 & \text{if } u_i = 0 \end{cases}$ (from Q.2 on homework (NLCP version))