...ng Duality Theorem

$X \subseteq \mathbb{R}^n$ convex

$f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}^m$ convex,

$h: \mathbb{R}^n \to \mathbb{R}^p$ affine (ie $h(x) = Ax - b$)

CQ holds:

~~$\exists\, x \in \text{int}\, X$~~

$\exists\, \hat{x} \in X$ s.t. $g(\hat{x}) < 0$, $h(\hat{x}) = 0$; θ

$0 \in \text{int}\, h(X)$. $h(X) = \{y : y = h(x) \text{ for some } x \in$

Then: $\inf \{f(x) : x \in X, g(x) \le 0, h(x) = 0\}$

$= \sup \{\theta(u,v) : u \ge 0\}$

Furthermore: If inf is finite then sup is achieved at $(\bar{u}, \bar{v})$

If inf is achieved at $\bar{x}$, then $\bar{u}^T g(\bar{x}) = 0$.

1. Let $\gamma = \inf \{ f(x) : x \in X, g(x) \leq 0, h(x) = 0 \}$.

If $\gamma = -\infty$, result trivial. (Weak duality)

Assume γ finite, let $\alpha(x) := f(x) - \gamma$.

Then the System : $\alpha(x) < 0$, $g(x) \leq 0, h(x) = 0$, $x \in X$

has no solution.

$\therefore \exists (u_0, u, v) \neq 0$ satisfying

(*) $$u_0 (f(x) - \gamma) + u^T g(x) + v^T h(x) \geq 0 \quad \forall x \in X,$$

$u_0 \geq 0, u \geq 0.$

Assume $u_0 = 0$:

By assumption, $\exists \hat{x}$ with $g(\hat{x}) < 0$, $h(\hat{x}) = 0$.

~~Thus, $u^T g$~~

We need $u^T g(\hat{x}) \geq 0$

$\therefore u = 0$, since $u \geq 0$, $g(\hat{x}) < 0$

$\therefore v^T h(x) \geq 0 \quad \forall x \in X$

But $0 \in \operatorname{int} h(X)$

$\therefore \exists x$ s.t. $h(x) = -\lambda v$ for some small +ve λ.

$\therefore v^T v = 0 \quad \therefore v = 0$ $\times$ to $(u_0, u, v) \neq 0$.

$\therefore u_0 > 0$

Divide (*) by u_0, write $\bar{u} = \frac{u}{u_0}$, $\bar{v} = \frac{v}{u_0}$:

$$f(x) + \bar{u}^T g(x) + \bar{v}^T h(x) \geq \gamma \quad \forall x \in X$$

$\therefore \sup\left(\theta(u,v) : u \geq 0\right) \geq \gamma$

But by weak duality, $\sup\left(\theta(u,v) : u \geq 0\right) \leq \gamma$

$\therefore \sup\left(\theta(u,v) : u \geq 0\right) = \gamma$,

and sup is achieved by $\bar{u}, \bar{v}$.

If inf is achieved at $\bar{x}$:

Then $f(\bar{x}) = \gamma$, $g(\bar{x}) \leq 0$, $h(\bar{x}) = 0$

But also,

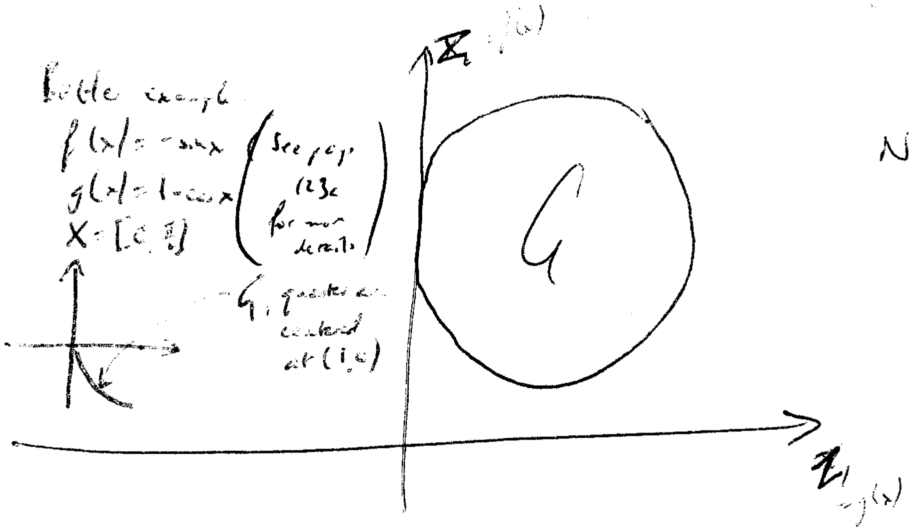
$$f(\bar{x}) + \bar{u}^T g(\bar{x}) + \bar{v}^T h(\bar{x}) \geq \gamma$$

$\therefore \bar{u}^T g(\bar{x}) \geq 0$

$\therefore \bar{u}^T g(\bar{x}) = 0$ //.

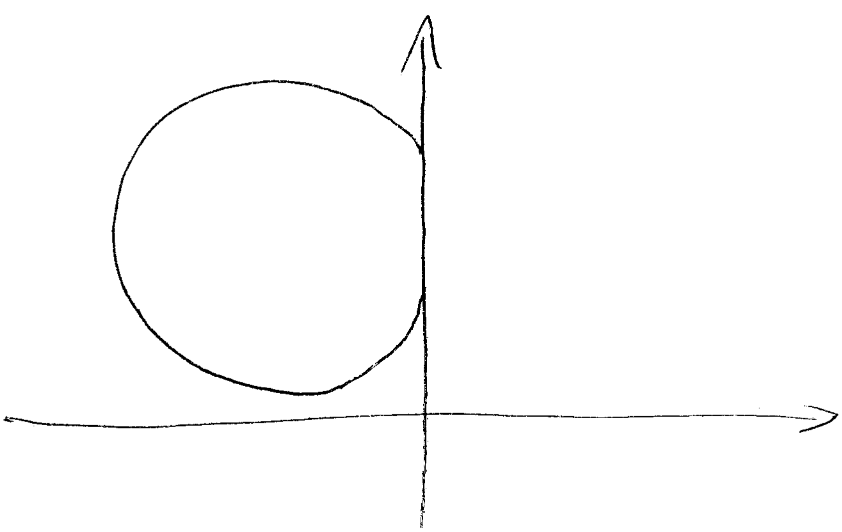
Geometrically, why do we need $g(x) < 0$ for some x, and $0 \in \text{int } h(X)$?

If $g(x) \geq 0 \quad \forall x \in X$:



Need to take x_2 axis as supporting hyperplane. But this corresponds to $u = \infty$, so sup is not nec. achieved

Similarly if $0 \notin \text{int } h(X)$, then



Same argument works, but now have $v \to -\infty$ giving optimal dual solution.

What does $0 \in \text{int}\, h(X)$ mean?

~~$\exists\, x_{int} \in X$~~

$\exists$ $\bar{x}, \hat{x} \in X$ with $A\bar{x} < b < A\hat{x}$.

If $X = \mathbb{R}^n$:

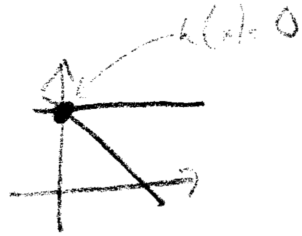
Let $y \in \mathbb{R}^m$,

let $x = A^T(AA^T)^{-1}(y+b)$

Then $Ax - b = (y+b) - b = y$

So for any $y \in \mathbb{R}^m$, $\exists\, x \in X$ s.t. $h(x) = y$.

$\therefore\ 0 \in \text{int}\, h(X)$.



If $X = \mathbb{R}^n_+$

$0 \in \text{int}\, h(X)$ roughly corresponds to the existence of a degenerate bfs.

Eg. $h(x) = \begin{pmatrix} x_1 + x_2 - 2 \\ x_2 - 2 \end{pmatrix}$ $x \in \mathbb{R}^2_+$, $0 = h\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Then $0 \notin \text{int}(h(X))$. $\begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \neq \cancel{=} h(x)$ for any $\varepsilon > 0$ $\begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} = h\begin{pmatrix} -\varepsilon \\ 2+\varepsilon \end{pmatrix}$

$\begin{pmatrix} -\varepsilon \\ 0 \end{pmatrix} \neq h(x)$ for any $\varepsilon > 0$ $\begin{pmatrix} -\varepsilon \\ 0 \end{pmatrix} = h\begin{pmatrix} -\varepsilon \\ 2 \end{pmatrix}$

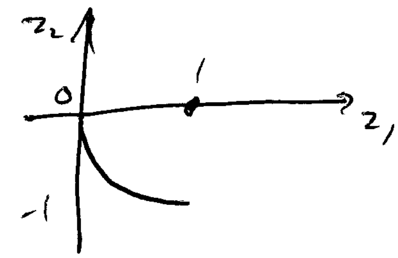
If $h(x) = \begin{pmatrix} x_1 + x_2 - 2 \\ x_2 - 1 \end{pmatrix}$ $x \in \mathbb{R}^2_+$

then $0 \in \text{int}(h(X))$ $0 = h\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{aligned} \min \quad & -\sin x \\ \text{s.t.} \quad & 1-\cos x \le 0 \\ & x \in [0, \tfrac{\pi}{2}] \end{aligned}$$

Optimal: $x = \frac{\pi}{2}$, value $= -1$.

This is only feasible solution.



$z_1 = 1-\cos x$, $z_2 = -\sin x$,

so $(z_1 - 1)^2 + z_2^2 = 1$

$x = 0$ gives $z = (0,0)$

$x = \frac{\pi}{2}$ gives $z = (1,1)$

Dual:

$$L(x,u) = -\sin x + u(1-\cos x)$$

$$\frac{d}{dx} L(x,u) = -\cos x + u \sin x, \quad \text{so} \begin{cases} u \tan x = 1 & \text{if } u > 0 \\ \cos x = 0 & \text{if } u = 0 \end{cases}$$

$\sup \theta(u) = 0$, but $\theta(u) < 0 \quad \forall u \geq 0$.