Example:

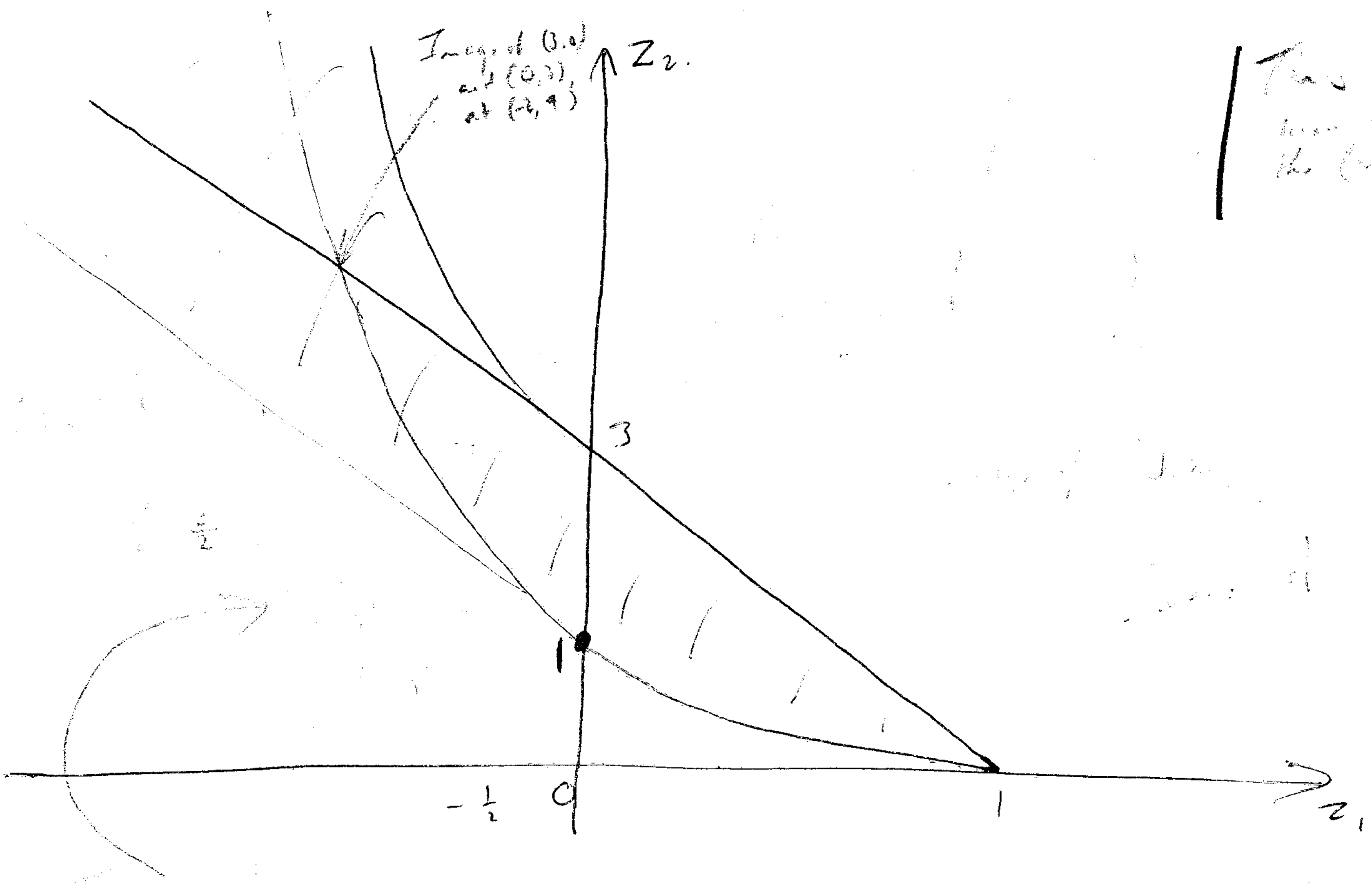
min $\quad x_1^2 + 3x_2$

$$-x_1 - x_2 + 1 \le 0$$

$$x_1, x_2 \ge 0.$$

$$(0, x_2) \to (1-x_2, 3x_2)$$

$$(x_1, 0) \to (1-x_1, x_1^2)$$



$\theta(3)$: gives optimal.

$\theta(u)$: ~~gives~~ crosses ~~at~~ between 0 and 1 if $u \le 3$

crosses at $-\infty$ if $u > 3$

Eg of Lagrangian Duality

$$\min \quad f(x) := x_1^2 + 3x_2$$
$$\text{s.t.} \quad g(x) := -x_1 - x_2 + 1 \leq 0$$
$$x \in X = \{x_1 \geq 0,\ x_2 \geq 0\}.$$

Lagrangian is $L(x,u) = x_1^2 + 3x_2 + u(-x_1 - x_2 + 1)$
$$= (x_1^2 - ux_1) + (3-u)x_2 + u$$

To find $\theta(u)$, we need to solve the problem

$\min_{x \text{ in } X} L(x,u)$ — L is convex, and X is convex. Thus, any turning point is a local min.

Now, $\frac{\partial L}{\partial x_1} = 2x_1 - u$, so want $x_1 = \frac{1}{2}u$

and $\frac{\partial L}{\partial x_2} = 3 - u$, so if $u > 3$, get $\theta(u) = -\infty$
$u < 3$, take $x_2 = 0$
$u = 3$, take $x_2 =$ anything.

Thus,
$$\theta(u) = \begin{cases} +\frac{1}{4}u^2 - \frac{1}{2}u^2 + u & \text{if } u \leq 3 \\ -\infty & \text{if } u > 3 \end{cases}$$
$$= \begin{cases} u - \frac{1}{4}u^2 & \text{if } u \leq 3 \\ -\infty & \text{if } u > 3. \end{cases}$$

Maximized when $\bar{u} = 2$, with $\theta(\bar{u}) = 1$.

Corresponding x is $\bar{x}_1 = \frac{1}{2}\bar{u} = 1$, $\bar{x}_2 = 0$.

Note that $g(\bar{x}) = 0$, and $f(\bar{x}) = 1$, so by weak duality $\bar{x}$ is optimal.

Then $L(\bar{x},\bar{u}) = \bar{x}_1{}^2 + 3\bar{x}_2 + \bar{u}\,(-\bar{x}_1 - \bar{x}_2 + 1) = 1.$

If $x \geq 0$,

$$L(x,\bar{u}) = x_1{}^2 + 3x_2 + 2(-x_1 - x_2 + 1)$$
$$= x_1{}^2 - 2x_1 + x_2 + 2$$
$$\geq -1 + 0 + 2 = 1 = L(\bar{x},\bar{u})$$

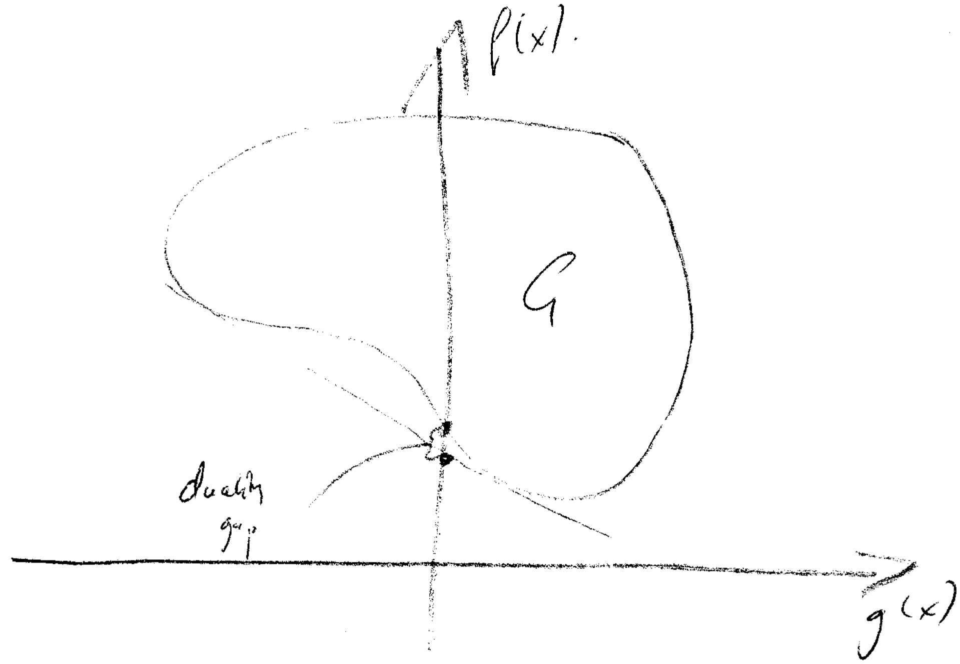
If $u \geq 0$:

$$L(\bar{x},u) = 1 + u(0) = 1 \leq L(\bar{x},\bar{u}).$$

So $\bar{x},\bar{u}$ is a saddle point.

Duality gap

Gap between optimal primal and optimal dual values.

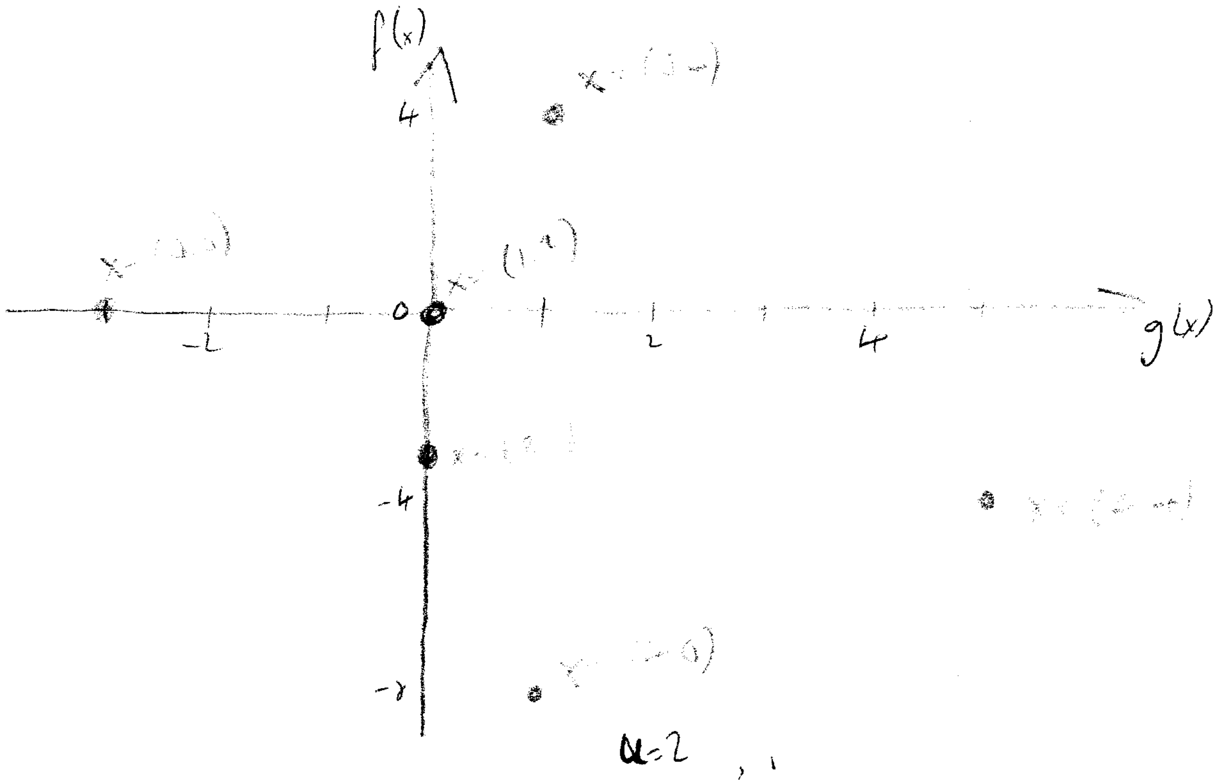


Example with a duality gap:

$$\begin{aligned} \text{Minimize} \quad & -2x_1 + x_2 \\ & x_1 + x_2 - 3 \leq 0 \\ & (x_1, x_2) \in X \end{aligned}$$

where $X = \{(0,0), (0,4), (4,4), (4,0), (1,2), (2,1)\}$.

$f(x) = -2x_1 + x_2$	0	4	-4	-8	0	-3
$g(x) = x_1 - x_2 - 3$	-3	1	5	1	0	0
	(0,0)	(0,4)	(4,4)	(4,0)	(1,2)	(2,1)



$$\theta(u) = \begin{cases} -4+5u & u \le -1 \\ -8+u & -1 < u \le 2 \\ -3u & 2 \le u \end{cases}$$

Dual value $= -6$.

Primal value $= -3$. - achieved by $(2,1)$.

Want conditions on when duality gap is zero.

First need the Lemma:

<u>Lemma</u> $X \subseteq \mathbb{R}^n$, convex, $X \neq \phi$.

$\alpha : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ convex

$h : \mathbb{R}^n \to \mathbb{R}^p$ affine, ie $h(x) = Ax - b$.

System 1: $\alpha(x) < 0$, $g(x) \leq 0$, $h(x) = 0$ for <u>some</u> $x \in X$

System 2: $u_0 \alpha(x) + u^T g(x) + v^T h(x) \geq 0$ for all $x \in X$

$(u_0, u) \geq 0$, $(u_0, u, v) \neq 0$

If System 1 has no solution x, then System 2 has a solution (u_0, u, v). The converse holds if $u_0 > 0$.

<u>Proof</u> Suppose System 1 has no solution.

Define $\Lambda = \{(p, q, r) : p > \alpha(x), q \geq g(x), r = h(x)$ for <u>some</u> $x \in X\}$

Λ is convex (Exercise)

System 1 has no solution. $\therefore (0,0,0) \notin \Lambda$

$\therefore \exists$ separating hyperplane separating $(0,0,0)$ from $\mathrm{cl}\,\Lambda$, ie $\exists u_0, u, v$ not all zero

s.t. $u_0 p + u^T q + v^T r \geq 0 \quad \forall (p,q,r) \in \mathrm{cl}\,\Lambda$

Fix an $x \in X$. Then p & q can be made arbitrarily large.

$\therefore (u_0, u) \geq 0$. Also, $(p, q, r) = [\alpha(x), g(x), h(x)]$ belongs to $\mathrm{cl}\,\Lambda$.

~~Therefore~~ The result follows.

verse:

Assume System 2 has a solution with $u_0 > 0$:

If $x \in X$ with $g(x) \leq 0$, $h(x) = 0$ then

$u_0 \cancel{f} \alpha(x) \geqslant 0$ since $u \geqq 0$

Thus $\cancel{\alpha}$ $\alpha(x) \geqslant 0$, so System 1 has no solution.