Lagrangian Dual Problem

Primal problem P:

$$\min \; f(x)$$

$$\text{s.t.} \quad g_i(x) \leq 0 \qquad i=1,\dots,m$$

$$h_j(x) = 0 \qquad j=1,\dots,p$$

$$x \in X \quad \text{some set } X. \text{ — eg: } X=\{x: 0\leq x_i\leq 1\},\ X=\mathbb{R}^n.$$

Define Lagrangian:

$$L(x,u,v) = f(x) + \sum u_i g_i(x) + \sum v_j h_j(x)$$

$$= f(x) + u^T g(x) + v^T h(x)$$

$$g: \mathbb{R}^n \to \mathbb{R}^m, \quad h: \mathbb{R}^n \to \mathbb{R}^p.$$

$$u \in \mathbb{R}^m, \; v \in \mathbb{R}^p, \; x \in \mathbb{R}^n.$$

Define Lagrangian dual function:

$$\theta(u,v) = \inf_{x\in X} \{L(x,u,v)\}.$$

Lagrangian dual problem:

$$\max \; \theta(u,v)$$

$$\text{s.t.} \quad u \geq 0.$$

Weak duality theorem:

Let $\bar{x}$ be feasible for (P), let $(\bar{u}, \bar{v})$ be feasible for the dual problem. Then $f(\bar{x}) \geq \theta(\bar{u}, \bar{v})$.

Proof

$$\theta(\bar{u},\bar{v}) = \inf_{x \in X} \{ f(x) + \bar{u}^T g(x) + \bar{v}^T h(x) \}$$

$$\leq f(\bar{x}) + \underbrace{\bar{u}^T}_{\geq 0} \underbrace{g(\bar{x})}_{\leq 0} + \bar{v}^T \underbrace{h(\bar{x})}_{=0}$$

$$\leq f(\bar{x})$$

//.

Corollary 1:

~~$\inf \{ f(x) + u^T g(x) +$~~

$$\inf \{ f(x) : g(x) \leq 0, h(x) = 0, x \in X \}$$

$$\geq \sup \{ \theta(u,v) : u \geq 0 \}.$$

Corollary 2:

If $f(\bar{x}) \leq \theta(\bar{u}, \bar{v})$, $\bar{x}$ primal feasible, $(\bar{u}, \bar{v})$ dual feasib then $\bar{x}$ and $(\bar{u}, \bar{v})$ solve primal, dual problems resp.

Cor 3: If $\inf \{ f(x) : g(x) \leq 0, h(x) = 0, x \in X \} = -\infty$ then

$$\theta(u,v) = -\infty \quad \forall u \geq 0, \forall v$$

Cor 4: If $\sup \{ \theta(u,v) : u \geq 0 \} = +\infty$, then the primal problem has no feasible solution.

EXAMPLE:

$$\min \quad x_1^2 + 3x_2$$

$$\text{s.t.} \quad -x_1 - x_2 + 1 \leq 0$$

$$(x_1, x_2) \cancel{\geq (0,0)} \in X = \{x_1 \geq 0, x_2 \geq 0\}.$$

$$L(x,u) = x_1^2 + 3x_2 + u(1 - x_1 - x_2)$$

$$\inf_{x \geq 0} \{x_1^2 + 3x_2 + u - ux_1 - ux_2\} = \inf_{x \geq 0} \{(x_1^2 - ux_1) + (3-u)x_2 + u\}$$

KKT:
$$2x_1 - u - v_1 = 0$$
$$3 - u - v_2 = 0$$

$$v_1 x_1 = 0, \quad v_2 x_2 = 0, \quad v_1, v_2 \geq 0.$$

$\Rightarrow \quad v_2 = 3 - u$, provided $u \leq 3$

$v_1 = 2x_1 - u \Rightarrow 0 = 2x_1^2 - x_1 u$

$\Rightarrow x_1 = 0$ or $\frac{u}{2}$.

So $x_1 = \frac{u}{2}$, $x_2 = \begin{cases} 0 & u \leq 3 \\ +\infty & u > 3. \end{cases}$

$$\Rightarrow \theta(u) = \begin{cases} u - \frac{u^2}{4} & u \leq 3 \\ -\infty & u > 3. \end{cases}$$

So $\max\limits_{0 \leq u \leq 3} \; u - \frac{u^2}{4}$ $\qquad u - \frac{u^2}{2} = -\frac{1}{4}(u-2)^2 + \frac{1}{2}$

So $\bar{u} = 2$, so dual value $= 2 - 1 =$

$\rightarrow \bar{x}_1 = 1, \bar{x}_2 = 0 \rightarrow$ primal value $= 1$

Solve by KKT conditions. 110B.

min $x_1^2 + 3x_2$

$-x_1 - x_2 + 1 \le 0$ (2) u_1

$-x_1 \le 0$ u_2

$-x_2 \le 0$ u_3

$2x_1 - \cancel{3}u_1 - u_2 = 0$

$+3 - u_1 - u_3 = 0$

$u_1(1-x_1-x_2) = 0$ (3), $u_2 x_1 = 0$, $u_3 x_2 = 0$.

$x_2 > 0 \Rightarrow u_3 = 0 \Rightarrow u_1 = 3$ (1) $\Rightarrow x_1 = 1 - x_2$ $\cancel{\Rightarrow u_1 \ge}$

Also $u_2 = 2x_1 - u_1 \Rightarrow 2x_1^2 - u_1 x_1 = 0 \Rightarrow x_1 = 0$ or $\frac{u_1}{2}$.

$x_1 = 0 \Rightarrow$ $\cancel{x_2}$ $u_1 = u_2 = 0$ $\#$ to (1).

$\therefore x_1 = \frac{u_1}{2}$. By (1), $u_1 = 3$ $\therefore$ $x_1 = \frac{3}{2}$ $\#$ to (2).

$\therefore x_1 = \frac{u_1}{2}$, $x_2 = 0$ $\therefore$ $\cancel{x_1}$ $u_1 > 0 \Rightarrow x_1 = 1$ by (3)

$\Rightarrow u_1 = 2$.