Duality in NLP.

~~Define~~

$$\min \; f(x)$$
$$g_i(x) \leq 0 \qquad i=1,\dots,m \qquad \text{(NLP)}$$

Define Lagrangian

$$L(x,u) = f(x) + \sum_{i=1}^{m} u_i \, g_i(x)$$

Classical Wolfe dual:

$$\begin{aligned} \max \quad & L(x,u) \\ & D_x L(x,u) = 0 \\ & u \geq 0 \end{aligned} \quad \equiv \quad \begin{aligned} \max \quad & f(x) + \sum_{i=1}^{m} u_i g_i(x) \\ & Df(x) + \sum_{i=1}^{m} u_i \, Dg_i(x) = 0 \\ & u \geq 0. \end{aligned}$$

Theorem $\left.\begin{array}{l} \bar{x} \text{ feasible for primal} \\ (\hat{x},\hat{u}) \text{ feasible for dual} \\ f, g_i \text{ convex} \end{array}\right\} \quad f(\bar{x}) \geq L(\hat{x},\hat{u})$

Proof

$$\begin{aligned} f(\bar{x}) &\geq f(\hat{x}) + Df(\hat{x})^T(\bar{x}-\hat{x}) \\ &= f(\hat{x}) - \left[\sum_{i=1}^{m} \hat{u}_i \, Dg_i(\hat{x})^T(\bar{x}-\hat{x})\right] \\ &\geq f(\hat{x}) - \sum_{i=1}^{m} \underbrace{\hat{u}_i}_{\geq 0} \, \underbrace{(g_i(\bar{x}) - g_i(\hat{x}))}_{\leq 0} \\ &\geq f(\hat{x}) + \sum_{i=1}^{m} \hat{u}_i \, g_i(\hat{x}) \\ &= L(\hat{x},\hat{u}) \end{aligned}$$

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Theorem Let $\bar{x}$ be optimal for (NLP). Assume f, g_i convex.
If CQ holds then $\exists\, \bar{u}$ such that $(\bar{x}, \bar{u})$ optimal for dual
and $$f(\bar{x}) = L(\bar{x}, \bar{u})$$

Proof CQ holds

$\therefore \exists\, \bar{u}$ s.t.
$$\nabla f(\bar{x}) + \sum_{i=1}^{m} \bar{u}_i \nabla g_i(\bar{x}) = 0$$
$$\sum_{i=1}^{m} \bar{u}_i g_i(\bar{x}) = 0$$
$$\bar{u}_i \geq 0.$$

$\therefore$ $\bar{u}_i$ is feasible for dual, and
$$L(\bar{x}, \bar{u}) = f(\bar{x}) + \sum_{i=1}^{m} \bar{u}_i g_i(\bar{x}) = f(\bar{x})$$

~~From previous theorem,~~

$\therefore$ if $(\hat{x}, \hat{u})$ feasible for dual, then
$$L(\hat{x}, \hat{u}) \leq f(\bar{x}) \qquad \text{from previous theorem}$$
$$= L(\bar{x}, \bar{u})$$

So ~~$\bar{x}$~~ $(\bar{x}, \bar{u})$ optimal for dual.

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What is Wolfe dual of

$$\min \quad \tfrac{1}{2} x^T C x - d^T x \qquad (P)$$
$$A x \leq b$$
$$x \geq 0 \qquad ?$$

Dual is:

$$\max \quad \tfrac{1}{2} x^T C x - d^T x + u^T (Ax - b) - w^T x$$
$$\text{s.t.} \quad Cx - d + A^T u - w = 0$$
$$u \geq 0, \quad w \geq 0.$$

ie
$$\max \quad -\tfrac{1}{2} x^T C x - b^T u + \left(x^T C x - d^T x + x^T A^T u - w^T x \right)$$
$$\text{s.t.} \quad Cx - d + A^T u - w = 0$$
$$u \geq 0, \quad w \geq 0$$

ie
$$\max \quad -\tfrac{1}{2} x^T C x - b^T u$$
$$\text{s.t.} \quad Cx + A^T u \geq d$$
$$u \geq 0.$$

For LP, the classical dual is the LP dual.

An example where dual optimal soln does not give primal optimal:

$$\min \quad x_1^2 - x_1 \qquad \left(= (x_1 - \tfrac{1}{2})^2 - \tfrac{1}{4} \right)$$

$$\text{s.t.} \quad x_1 - x_2 \leq 0 \qquad (P).$$

$$x_1, x_2 \geq 0$$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \; d = \begin{pmatrix} +1 \\ 0 \end{pmatrix}, \; A = (1, -1)$$

(CD):

$$\max \quad -x_1^2$$

$$\text{s.t.} \quad \begin{pmatrix} 2x_1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 1 \\ -1 \end{pmatrix} \geq \begin{pmatrix} +1 \\ 0 \end{pmatrix}$$

$$u \geq 0.$$

Consider $\hat{x} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$, $u = 0$:

Feas for (CD), with value $-\frac{1}{4}$

~~Infeas for~~ $\hat{x}$ infeas for (P)

Consider $\bar{x} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$: Feas for (P), with value $-\frac{1}{4}$.

$\therefore (\hat{x}, \hat{u})$ optimal for (CD), but leads to an infeas point for ~~D~~ (P).

Mangasarian - Ponstein Converse Theorem (1965)

Let f and each g_i be differentiable and convex. If $(\hat{x}, \hat{u})$ is optimal for the classical (Wolfe) dual and if $L(x, \hat{u})$ is strictly convex at $\hat{x}$, then $\hat{x}$ is optimal for the primal and $f(\hat{x}) = L(\hat{x}, \hat{u})$.

Proof Lengthy (see Jour. Math. Analysis & Appl. Vol 11, 1965). //

In example, L is not strictly convex, because ~~it can change x_2 arbitrarily and not change obj. fn value.~~

$$L(x,u,w) = x_1^2 - x_1 + u_1(x_1 - x_2) - w_1 x_1 - w_2 x_2 ,$$

so only linear in x_2 for a fixed u, w.