Sufficiency theorem

Theorem If:

i) the feasible region S for (NLP) is compact

ii) CQ holds throughout the feasible region

iii) $\bar{x}$ is the unique feasible solution to the KKT condns

then

$\bar{x}$ is the global minimizer of f over S.

Proof

S is compact, so f achieves its bounds over S. (Weierstrass Thm)

Therefore, $\exists$ global minimizer $\hat{x}$ of f over S.

CQ holds at $\hat{x}$.

$\therefore$ the KKT condns hold at $\hat{x}$.

$\therefore$ $\hat{x} = \bar{x}$, ie $\bar{x}$ is the global minimizer of f over S //

Note:

If $\bar{x}$ strictly ~~positive~~ feasible, and $\bar{x}$ is a local minimizer ~~&~~ and CQ satisfied at $\bar{x}$ then KKT condns become $\nabla f(\bar{x}) = 0$

KKT condns for LP

Consider

$$\min \; c^T x$$
$$Ax = b$$
$$x \geq 0$$

Then $h_i(x) = b_i - \sum a_{ij} x_j$ $\quad g_i(x) = -x_i \quad f(x) = c^T x$

So

$$\nabla f(x) = c$$

$$\nabla g_i(x) = -e_i \text{ - the } i\text{th unit vector } \quad b$$

$$\nabla h_i(x) = -a_{i\cdot} \quad \text{the } i\text{th row of } A$$

So KKT:

$\exists$ u, v s.t.

$$c - \sum u_i e_i - \sum v_j a_{j\cdot} = 0$$

$$u_i x_i = 0 \quad \forall i$$
$$u_i \geq 0$$

c $\quad \exists\, u, v$ s.t.

$$c - u - A^T v = 0$$

$$x_i u_i = 0$$

$$u \geq 0.$$

i.e. $\exists\, u, v$ s.t.

$$A^T v \leq c \qquad \text{Dual feasibility}$$

$$x^T (c - A^T v) = 0 \qquad \text{Complementary slackness.}$$

Notice that (CQ):

feasible directions at $\bar{x}$ are

$$\{d : Ad = 0,\ d_i \geq 0 \text{ if } x_i = 0\} = T.$$

Also,

$$H_0 \cap G' = \{d : Dh_j^*(\bar{x})^T d = 0,\ Dg_i(\bar{x})^T d \leq 0 \quad i \in I\}$$

$$= \{d : Ad = 0,\ -d_i \leq 0 \text{ for } i \in I\}$$

So $T = H_0 \cap G'$.

<u>Claim</u> If (NLP) linearly constrained,

CQ holds throughout feasible region.

KKT ~~suff~~ Sufficient Conditions (Ineq constraints)

$$\min \; f(x)$$
$$\text{s.t.} \; g_i(x) \le 0 \quad i=1,\dots,m \qquad \text{(NLP)}.$$

Let f, g_i be convex. If $\bar{x}$ is feasible and if $\exists\, \bar{u} \ge 0$ s.t.

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) = 0 \quad \text{and} \quad \bar{u}_i g_i(\bar{x}) = 0 \quad \forall i$$

then $\bar{x}$ is a minimizing point.

Pf For any feasible x,

$$\begin{aligned} f(x) - f(\bar{x}) &\ge \nabla f(\bar{x})^T (x - \bar{x}) \qquad \text{Support inequality} \\ &= -\sum_{i=1}^{m} \bar{u}_i [\nabla g_i(\bar{x})^T (x - \bar{x})] \\ &\ge -\sum_{i=1}^{m} \bar{u}_i [g_i(x) - g_i(\bar{x})] \\ &= -\sum_{i=1}^{m} \bar{u}_i g_i(x) \\ &\ge 0 \qquad \text{since } \bar{u}_i \ge 0,\ g_i(x) \le 0. \end{aligned}$$

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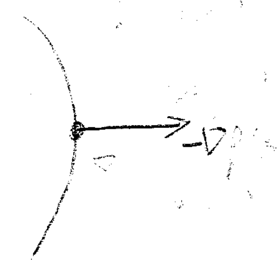

KKT Sufficient conditions (Eq constraints)

$$\min f(x)$$
$$g_i(x) \le 0 \qquad i=1,\dots,m$$
$$h_j(x) = 0 \qquad j=1,\dots,p$$

Let $\bar{x}$ be feasible and assume $\exists\, \bar{u}, \bar{v}$ s.t.

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} \bar{u}_i \nabla g_i(\bar{x}) + \sum_{j=1}^{p} \bar{v}_j \nabla h_j(\bar{x}) = 0$$

and $\quad u_i g_i(\bar{x}) = 0, \quad u_i \ge 0 \quad \forall i$

Further, assume: f, g_i are convex, $\bar{v}_j > 0 \Rightarrow h_j(\bar{x})$ convex (Let $J = \{j : \bar{v}_j >$
$\bar{v}_j < 0 \Rightarrow h_j(\bar{x})$ concave (Let $K = \{j : \bar{v}_j < 0$

Then $\bar{x}$ is a ~~local~~ minimizing point.

Proof Let x be feasible. Then

$$f(x) - f(\bar{x}) \ge \nabla f(\bar{x})^T (x - \bar{x})$$
$$= -\left[\sum_{i=1}^{m} \bar{u}_i \nabla g_i(\bar{x})^T (x-\bar{x}) + \sum_{j=1}^{p} \bar{v}_j \nabla h_j(\bar{x})^T (x-\bar{x})\right]$$
$$\ge -\left[\sum_{i=1}^{m} \bar{u}_i \{g_i(x) - g_i(\bar{x})\} + \sum_{j\in J} \bar{v}_j (h_j(x) - h_j(\bar{x})) + \sum_{j\in K} \bar{v}_j (h_j(x) - h_j(\bar{x}))\right]$$
$$= -\sum_{i=1}^{m} \bar{u}_i g_i(x) \ge 0. \qquad (h_j(x) = h_j(\bar{x}) = 0)$$

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NB: Can relax to : level sets are convex / concave.

Necessity of condition on equality constraints:

$$\min \quad x_1^2 + x_2^2$$

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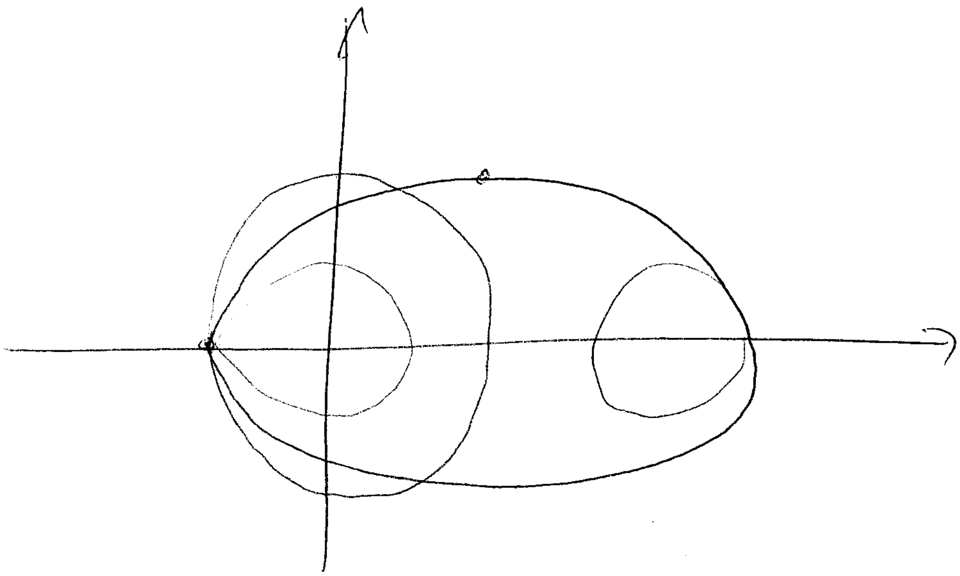
$$\text{s.t.} \quad (x_1 - 1)^2 + 4x_2^2 = 4$$



$$\nabla h = \begin{bmatrix} 2(x_1 - 1) \\ 8x_2 \end{bmatrix} \qquad \nabla f = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

At $(-1, 0)$: multiplier is $-\frac{1}{2}$.

h is convex and $\exists$ nearby better points.