Problems with equality constraints

$$\min \quad f(x)$$
$$\text{s.t.} \quad g_i(x) \le 0 \qquad i=1,\dots,m \qquad (NLP)$$
$$h_j(x) = 0 \qquad j=1,\dots,p$$

Fritz-John If $\bar{x}$ is a local min for (NLP) then $\exists$ ~~$\not\!\!\!$~~ u_0, $u \in \mathbb{R}^m$, ~~not all zero~~ $v \in \mathbb{R}^p$ not all zero

s.t.
$$u_0 \nabla f(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) + \sum_{j=1}^{p} v_j \nabla h_j(\bar{x}) = 0$$
$$u \ge 0, \quad \sum u_i g_i(\bar{x}) = 0.$$

CQ: Let $H_0 = \{d : \nabla h_j(\bar{x})^T d = 0,\ j=1,\dots,p\}$.
Constraint qualification is $T = G' \cap H_0$.

Karush-Kuhn-Tucker conditions

If $\bar{x}$ is a local min for (NLP) and CQ holds at $\bar{x}$, then $\exists\, u \in \mathbb{R}^m, v \in \mathbb{R}^p$ with
$$\nabla f(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) + \sum_{j=1}^{p} v_j \nabla h_j(\bar{x}) = 0$$
$$u_i g_i(\bar{x}) = 0 \quad \forall i.$$
$$u_i \ge 0 \quad \forall i.$$

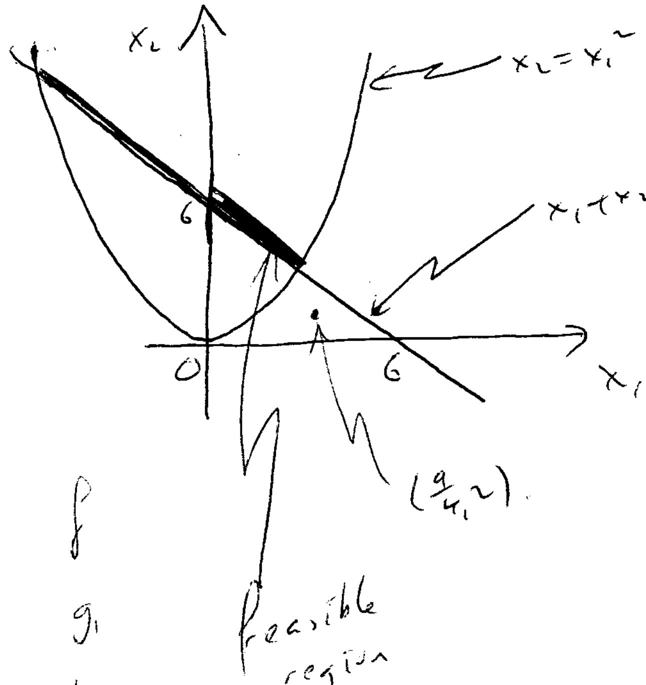
Example (Ex 4.7)

$$\min \quad (x_1 - \tfrac{9}{4})^2 + (x_2 - 2)^2$$

subj to $\quad x_2 - x_1^2 \geq 0$

$\quad x_1 + x_2 = 6$

$\quad x_1, x_2 \geq 0$



$\longleftrightarrow \min \quad (x_1 - \tfrac{9}{4})^2 + (x_2 - 2)^2 \qquad f$

s.t. $\quad x_1^2 - x_2 \leq 0 \qquad g_1$

$\quad +x_1 + x_2 - 6 = 0 \qquad h_1$

$\quad -x_1 \leq 0 \qquad g_2$

$\quad -x_2 \leq 0 \qquad g_3$

$$f(x) = (x_1 - \tfrac{9}{4})^2 + (x_2 - 2)^2$$

$$\cancel{\nabla f(x) = 2(x_1 - \tfrac{9}{4}) + 2(x_2 - 2)}$$

$$\cancel{\nabla g_1(x) = 2x_1}$$

$$\nabla f(x) = \begin{pmatrix} 2(x_1 - \frac{9}{4}) \\ 2(x_2 - 2) \end{pmatrix}$$

$$\nabla g_1(x) = \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} \qquad \nabla g_2(x) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \nabla g_3(x) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\nabla h_1(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\left(\nabla^2 g_1(x) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right.$ psd everywhere $\therefore$ convex $\left. \right)$.

KKT condns:

$$2\left(x_1 - \tfrac{9}{4}\right) + 2u_1 x_1 - u_2 \qquad + v_1 = 0$$

$$2\left(x_2 - 2\right) - u_1 \qquad - u_3 + v_1 = 0$$

$$u_1, u_2, u_3 \geqslant 0$$

$$u_1\left(x_1^2 - x_2\right) = 0$$

$$u_2 x_1 = 0, \quad u_3 x_2 = 0.$$

[illegible]

$x_1 = 2,\ x_2 = 4$: $\quad u_2 = u_3 = 0$

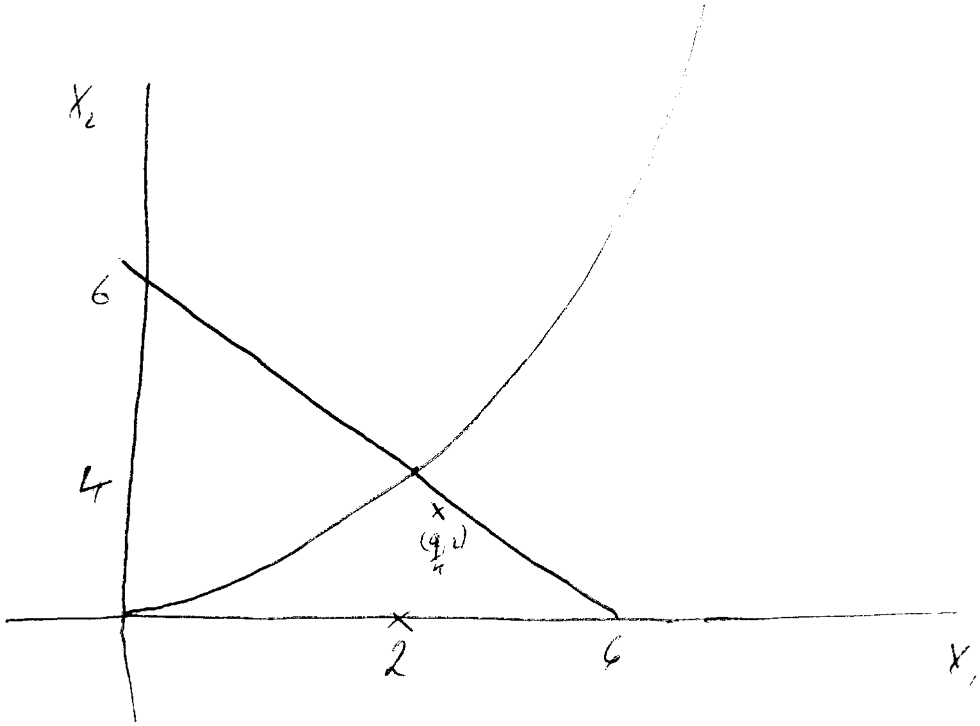
$$2\left(-\tfrac{1}{4}\right) + \cancel{2}4u_1 \qquad + v_1 = 0 \qquad (1)$$

$$2\,(2)\ \cancel{7} - u_1 \qquad + v_1 = 0 \qquad (2)$$

$$(1)-(2) \Rightarrow -4\tfrac{1}{2} + 5u_1 = 0$$

$$\Rightarrow u_1 = \tfrac{9}{10}, \ \cancel{v_1 =}$$

$$4(2)+(1) \Rightarrow 5v_1 = -15\tfrac{1}{2} \Rightarrow v_1 = -3\tfrac{1}{10}$$



X_2
6
4
(9/4, 2)
2
6
X_1

[illegible]

$$u_2 > 0 \Rightarrow x_1 = 0 \Rightarrow x_2 = 6 \Rightarrow u_3 = 0, u_1 = 0$$

So need
$$\left.\begin{aligned} 2\left(-\tfrac{9}{4}\right) - u_2 + v_1 &= 0 \\ 2\,(4) \qquad + v_1 &= 0 \end{aligned}\right\} \quad \cancel{v_1 = -8}$$

$$\Rightarrow v_1 = -8, \; u_2 = -12\tfrac{1}{2} < 0$$

So must have $u_2 = 0$.

$$u_3 > 0 \Rightarrow x_2 = 0 \Rightarrow x_1 = 6 \Rightarrow \cancel{x_1^2\; x_2} \; x_1^2 - x_2 > 0$$

So must have $u_3 = 0$.

$$u_1 = 0 \Rightarrow \begin{aligned} 2\left(x_1 - \tfrac{9}{4}\right) \quad + v_1 &= 0 \\ 2\left(x_2 - 2\right) \quad + v_1 &= 0 \end{aligned}$$

Add: $2\left(x_1 + x_2\right) - \frac{17}{2} + 2v_1 = 0$

ie $12 - \frac{17}{2} + 2v_1 = 0$ ie $v_1 = -\frac{7}{4}$

$$\left.\begin{aligned} &\Rightarrow x_1 = \frac{9}{4} + \frac{7}{8} = \frac{25}{8}, \; x_2 \; \cancel{\tfrac{7}{8} + 2} \\ &x_2 = \frac{7}{8} + 2 = \frac{23}{8} \end{aligned}\right\}$$ Closest point on the line $x_1 + x_2 = 6$ to the point $\left(\frac{9}{4}, 2\right)$.

$$\Rightarrow x_1^2 - x_2 > 0$$

So must have $u_1 > 0$, $u_2 = 0$, $u_3 = 0 \Rightarrow x_1^2 - x_2 = 0$, $x_1 + x_2 = 6$

$\Rightarrow x_1 = 2, \; x_2 = 4 \Rightarrow$