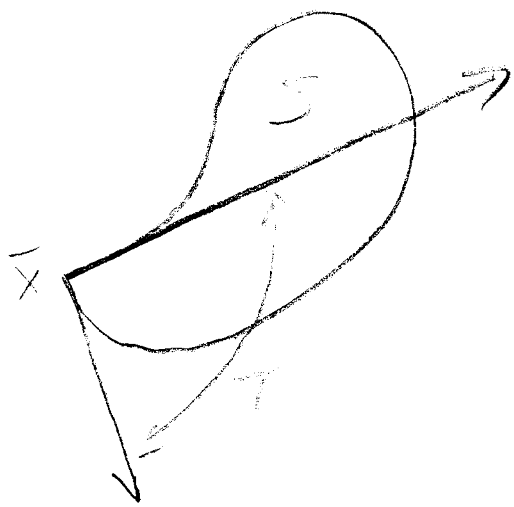
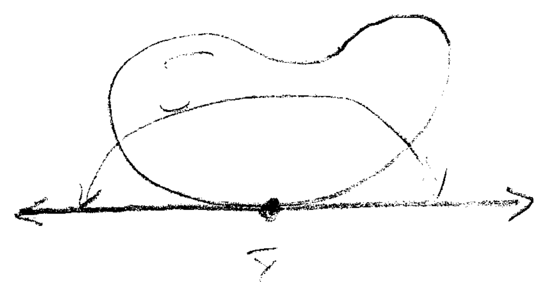
Defn $S \subseteq \mathbb{R}^n$, $\bar{x} \in \mathrm{cl}\, S$.

Cone of tangents T of S at $\bar{x}$, is the set of all directions d such that $d = \lim_{k\to\infty} \lambda_k (x_k - \bar{x})$, where $\lambda_k > 0, x_k \in S$ for each k, and $x_k \to \bar{x}$.





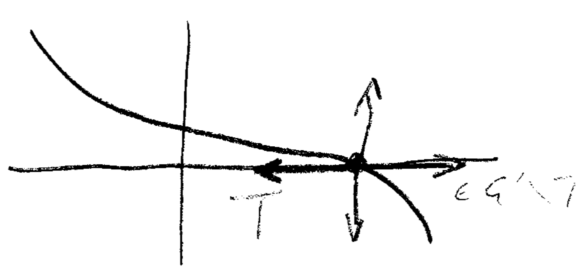
Let $G' = \{d : \nabla g_i(\bar{x})^T d \leq 0 \quad \forall i \in I\}$.

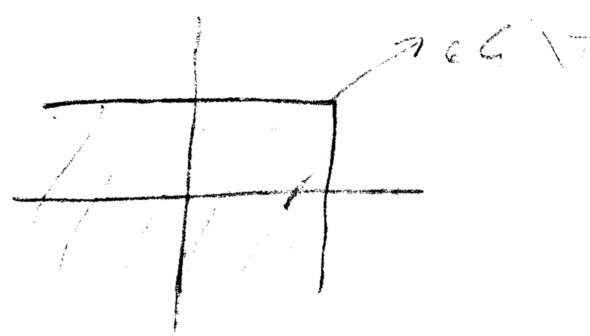
Note that $T \subseteq G'$.

Constraint Qualification:

$$T = G'$$

Previous examples:





__Theorem__ If $\bar{x}$ is a local minimum to (NLP) then $F_0 \cap T = \phi$,

where $F_0 = \{ d : \nabla f(\bar{x})^T d < 0 \}$.

__Proof__ Let $d \in T$.

$$d = \lim_{k \to \infty} \lambda_k (x_k - \bar{x})$$

$$\underset{\substack{\vee\!/ \\ 0}}{f(x_k) - f(\bar{x})} = \nabla f(\bar{x})^T (x_k - \bar{x}) + \|x_k - \bar{x}\| \alpha(\bar{x}, x_k - \bar{x})$$

So $d \notin F_0$, since $x_k \to \bar{x}$ //.

Karush Kuhn-Tucker theorem

If $\bar{x}$ is a local optimal solution to (NLP) and CQ holds, then

$\exists$ nonnegative scalars u_i for $i \in I$ such that

$$\nabla f(\bar{x}) + \sum_{i \in I} u_i \nabla g_i(\bar{x}) = 0.$$

Proof Recall Farkas Lemma:

(I) $\quad Ax = b$
$\quad\quad x \geq 0$

(II) $\quad A^T y \leq 0$
$\quad\quad b^T y > 0$

~~Now $\bar{x}$ is a local optimum, so~~ If $\nabla g_i(\bar{x})^T d \leq 0$ then $d \in G'$

Also $d \in T$. by CQ.

$\bar{x}$ is a local minimum, so $F_0 \cap T = \phi$

i.e. $d \notin F_0$ i.e. $d^T \nabla f(\bar{x}) \geq 0$.

$\therefore$ System (II) is infeasible

$\therefore$ System (I) has a solution, i.e. $\exists\, u \geq 0$ s.t.

$$\nabla f(\bar{x}) + \sum_{i \in I} u_i \nabla g_i(\bar{x}) = 0.$$

//

KKT condns in vector notation:

if $\bar{x}$ is a local minimum, and CQ holds, then $\exists\ \bar{u} \in \mathbb{R}^m$ with

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} \bar{u}_i \nabla g_i(\bar{x}) = 0$$

$$\bar{u}_i\, g_i(\bar{x}) = 0$$

$$\bar{u}_i \geq 0.$$

~~<u>Lemma</u> If $\{\nabla g_i \text{ and } i \in I\}$ are lin indep at $\bar{x}$, then the constraint qualification holds.~~

<u>Lemma</u> If $\exists\ \hat{d}$ s.t. $\hat{d}^T \nabla g_I < 0$ then CQ holds at $\bar{x} \in S$.

<u>Pf</u> Let $\bar{d} \in \cancel{D}$, and set $d_k = \bar{d} + \left(\frac{1}{k}\right)\hat{d}$. //

<u>Lemma</u> If $\{\nabla g_i : i \in I\}$ are lin indep, then constraint qualification holds.

<u>Proof</u> Consider the dual pair of LPs:

$$\max\ 0 \qquad\qquad \min\ -\sum w_i$$

$$\nabla g_i(\bar{x})^T d \leq -1 \quad \forall i \in I \quad (P) \qquad\qquad \sum w_i \nabla g_i(x) = 0 \quad (D)$$

$$w_i \geq 0.$$

Only soln to (D) is $w = 0$. //

Corollary (to first lemma on p. 95)

Slater Constraint Qualification:

If $g_i(x)$ convex $\forall i$ and if $\exists\ \hat{x}$ with $g_i(\hat{x}) < 0\ \forall i$, then CQ holds at all feasible x.

Proof Let $\bar{x}$ be feasible, with active set I.

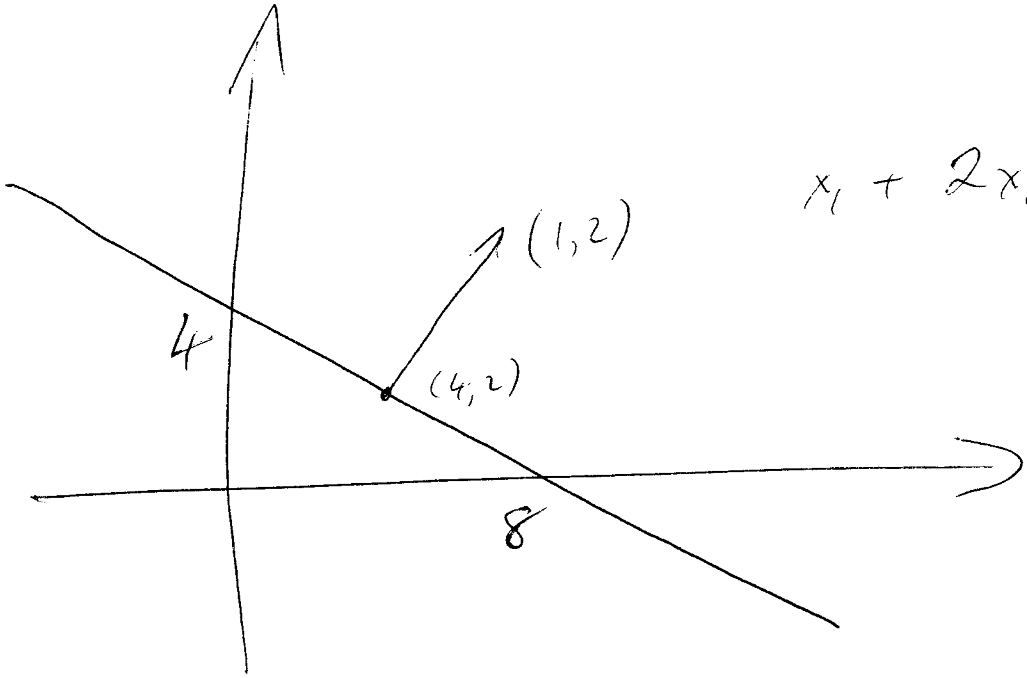
Let $\hat{d} = \hat{x} - \bar{x}$.

Then $g_i(\hat{x}) < 0 = g_i(\bar{x})\ \forall i \in I$

and $g_i(\hat{x}) \geq g_i(\bar{x}) + \nabla g_i(\bar{x})^T(\hat{x} - \bar{x})$

so $\nabla g_i(\bar{x})^T \hat{d} < 0 \quad \forall i \in I$

$\blacksquare$



$$x_1 + 2x_2 \le 8.$$

$$\min \quad (x_1-7)^2 + (x_2-8)^2$$
$$\text{s.t.} \quad x_1 + 2x_2 \le 8.$$

$$\boxed{\begin{array}{ll} \min & \tfrac{1}{2}x_1^2 - 7x_1 + \tfrac{1}{2}x_2^2 - 8x_2 \\ \text{s.t.} & x_1 + 2x_2 - 8 \le 0 \end{array}}$$

$$\nabla f = \begin{pmatrix} x_1 - 7 \\ x_2 - 8 \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

So need soln to

$$x_1 - 7 + u = 0 \quad ①$$
$$x_2 - 8 + 2u = 0 \quad ②$$
$$u(x_1 + 2x_2 - 8) = 0 \quad ③$$
$$x_1 + 2x_2 - 8 \le 0 \quad ④ \qquad u \ge 0. \quad ⑤$$

$u = 0 \Rightarrow x_1 = 7, \ x_2 = 8$

Violates $x_1 + 2x_2 - 8 \leq 0$

$u > 0 \Rightarrow x_1 + 2x_2 - 8 = 0$ ie $x_1 = 8 - 2x_2$

Substitute into ①:

$$1 - 2x_2 + u = 0$$
$$-8 + x_2 + 2u = 0$$

$$\therefore -15 + 5u = 0 \Rightarrow u = 3$$

$$\Rightarrow x_1 = 4, \ x_2 = 2. \quad \checkmark.$$