LP Duality

Diet Problem

Consumer has foods of type $j = 1, \ldots, n$ available on market at

prices c_j per unit,

and nutritional requirements for nutrients $i = 1, \ldots, m$ (vitamins, minerals, etc.)

b_i units required daily

a_{ij} = amount of ith nutrient present in one unit of jth food type

Solve:

$$\min \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t. } \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \forall i$$

$$x_j \geq 0 \quad \forall j$$

$\min c^T x$
s.t. $Ax \geq b$
$x \geq 0$

Find purchase amount x_j to minimize cost and meet nutritional requirements.

Related problem for producer of nutrient pills.

Produce pills of types $i = 1, \ldots, m$

Set unit prices y_i for pills so that consumer buys pills

$y_i \geq 0, \quad i = 1, \ldots, m$

Consider jth food type

Consumer has 2 options: (1) purchase 1 unit at cost c_j giving $\begin{bmatrix} a_{1j} \\ a_{mj} \end{bmatrix}$ nutr

(2) purchase nutrients from producer at cost

$\sum \cancel{y_i a_{ij}}$ $y_1 a_{1j} + \cdots + y_m a_{mj}$

To make pill prices attractive, make option (2) preferable to (1):

ie $\displaystyle\sum_{i=1}^{n} y_i a_{ij} \leq c_j$

Do this $\forall j$ to limit purchase to pills only

So, in order to maximize profit from pill sales to consumer, producer

solves

$$\max \sum_i b_i y_i$$
$$\text{s.t.} \quad \sum_{i=1}^{n} y_i a_{ij} \leq c_j \quad , 1 \leq j \leq n$$
$$y_i \geq 0 \quad , 1 \leq i \leq m.$$

ie

$$\max \quad b^T y$$
$$A^T y \leq c$$
$$y \geq 0$$

<u>General duality correspondence</u>

max	min
rhs	obj fcn
obj fcn	rhs
A	A^T
~~constr.~~ $\left\{\begin{matrix} \text{eq.} \\ \leq \\ \geq \end{matrix}\right\}$ constrs	~~vrbls~~ $\left\{\begin{matrix} \text{unres.} \\ \geq 0 \\ \leq 0 \end{matrix}\right\}$ ~~variables~~ variable
vrbls $\left\{\begin{matrix} \text{unres} \\ \geq 0 \\ \leq 0 \end{matrix}\right\}$	$\left\{\begin{matrix} = \\ \geq \\ \leq \end{matrix}\right\}$ constrs

eg: Dual of $\begin{matrix} \min & c^T x \\ & Ax = b \\ & x \geq 0 \end{matrix}$ (P) is $\begin{matrix} \max & b^T y \\ & A^T y \leq c \end{matrix}$ (D).

Weak duality theorem

~~If x is feasible in the minimization~~

If x is primal feasible and y is dual feasible then

$$c^T x \geq b^T y.$$

Proof $\quad b^T y = (Ax)^T y = \underbrace{x^T}_{\geq 0} \underbrace{(Ay)}_{\leq c} \leq c^T x \quad //$

Theorem

Sufficient Optimality Criterion:

If $\bar{x}$ is (P)-feas and $\bar{y}$ is (D)-feas and $c^T\bar{x} = b^T\bar{y}$ then

$\bar{x}$ is (P)-opt and $\bar{y}$ is (D)-optimal.

Proof $\quad$ For any (P)-feas x, we have $c^T x \geq b^T \bar{y} = c^T \bar{x}$ by hypothesis

~~Thus $c^T x$~~. $\quad //$

Provides an easy check of optimality:

If someone gives you an $\bar{x}$ and a $\bar{y}$ and claims they are optimal for (P) and (D) respectively, you can check

i) $\bar{x}$ is (P)-feasible

ii) $\bar{y}$ is (D)-feasible

iii) $c^T\bar{x} = b^T\bar{y}$.

Strong Duality Theorem

The following are mutually exclusive and exhaustive possibilities for (P) and (D):

(I) (P) and (D) infeas

(II) (D) infeas, (P) feasible (& unbdd in obj value)

(III) (P) infeas, (D) feas (& unbdd in obj value)

(IV) (P) & (D) feas and max = min.

Proof: If optimal value of (P) is finite: let $\bar{x} = B^{-1}b$ be optimal soln. (Prove
$\bar{y} = c_B B^{-1}$. (Assume A has full rank)

$\bar{r} = c - c_B B^{-1} A \geq 0$, so $\bar{y}$ is dual feasible.

Also, $b^T \bar{y} = c_B^T B^{-1} b = c^T \bar{x}$. ////

Complementary Slackness

A pair of primal and dual feasible solutions are optimal to the respective problems in a primal-dual pair of LP's, iff, whenever these feasible solutions make a slack variable in one problem strictly positive, the value of the associated non-negative variable of the other problem is zero.

Proof

We prove for ~~min~~

$$\min \ c^T x \qquad\qquad \max \ b^T y$$
$$Ax = b \qquad\qquad A^T y \le c.$$
$$x \ge 0$$

Theorem states:

$\cancel{\Rightarrow}$ $\hat{x}, \hat{y}$ optimal $\Leftrightarrow$ $\cancel{(\hat{x}_i > 0 \Leftrightarrow a_i^T y = c)}$

i) $\hat{x}_i > 0 \Rightarrow a_i^T \hat{y} = c_i$

and ii) $a_i^T \hat{y} < c_i \Rightarrow \hat{x}_i = 0$.

i.e. $\hat{x}, \hat{y}$ optimal $\Leftrightarrow$ $\hat{x}_i (c_i - a_i^T \hat{y}) = 0$

$\Rightarrow$: $c_i - a_i^T \hat{y} \ge 0$, $\hat{x}_i \ge 0$ by feasibility.

~~Also,~~ $\cancel{x^T(c - A^T y) =}$

$\therefore \hat{x}_i (c_i - a_i^T \hat{y}) \ge 0 \quad \forall i$

$\therefore 0 \le$ ~~Also,~~ $\sum_{i=1} \hat{x}_i (c_i - a_i^T \hat{y}) = \hat{x}^T (c - A^T \hat{y}) = c^T \hat{x} - (Ax)^T \hat{y}$

$= c^T \hat{x} - b^T \hat{y} = 0$ since $\hat{x}, \hat{y}$ optimal pair //.

$\Leftarrow$: $\hat{x}_i (c_i - a_i^T \hat{y}) = 0 \quad \forall i$

$\therefore \quad \hat{x}^T (c - A^T \hat{y}) = 0$

$\therefore \quad c^T \hat{x} = (A\hat{x})^T \hat{y} = b^T \hat{y}$ //

Farkas' Lemma

$A \in \mathbb{R}^{m \times n}$

Exactly one of the following systems is feasible:

(I): $Ax = b$
$x \geq 0$

(II): $A^T y \leq 0$
$b^T y > 0$

Proof Consider the primal LP

$$\min \quad 0 \cdot x \qquad (\hat{P})$$
$$Ax = b$$
$$x \geq 0$$

~~Its~~ Its dual is

$$\max \quad b^T y \qquad (\hat{D})$$
$$A^T y \leq 0$$

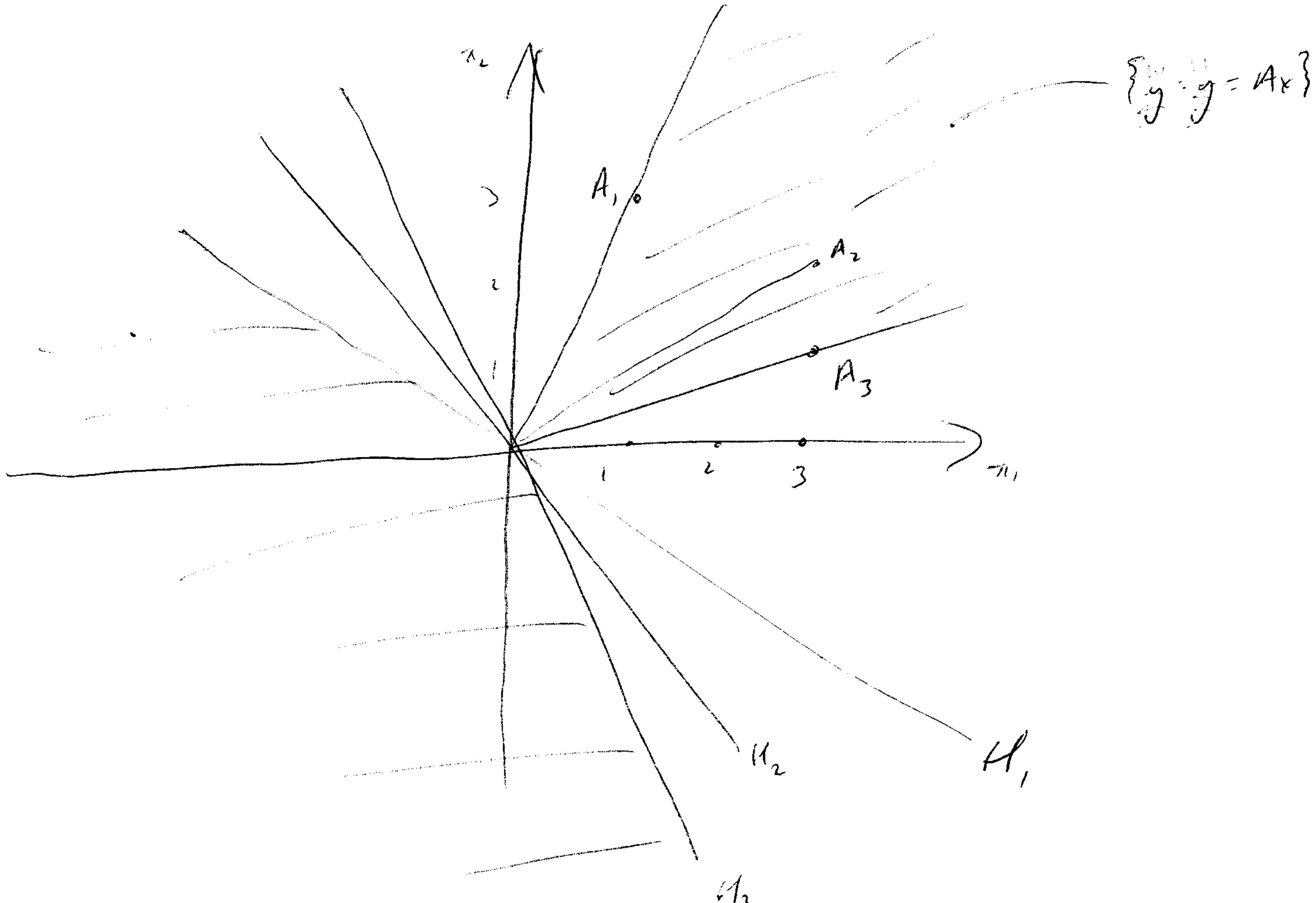
Assume ~~if~~ $(\hat{P})$ has a feasible solution, it has optimal value 0.

optimal value of $(\hat{D})$ is zero $\Rightarrow$ if $A^T y \leq 0$ then $b^T y \leq 0$.

Note also, $(\hat{D})$ always feasible (take $y = 0$) //

Geometrical illustration:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$



$\pi A \le 0$

If $b \underset{(\notin)}{\in}$ cone$\{A_1, A_2, A_3\}$ then

b makes [illegible] $\underset{(>)}{\le} 0$ [illegible]

[illegible]