<u>Defn</u> A <u>homogeneous solution</u> corresponding to (P) is a vector y satisfying

$$Ay = 0$$
$$y \geq 0.$$

Exercise: The set of all homogeneous solutions is a convex cone.

If $\tilde{x}$ is feasible for (P) and $\hat{y}$ is a homog. soln then $\tilde{x} + \theta\hat{y}$ is feasible in (P) $\forall \theta \geq 0$.

<u>Resolution Theorem</u>

Every feasible solution of (P) can be expressed as the sum of

(i) a convex combination of the BFS's

and (ii) a homogeneous solution corresponding to (P).

__Proof__ Suppose $\tilde{x}$ is a feasible solution of (P).

Let J be support of $\tilde{x}$, so

$$J = \{ j : \tilde{x}_j > 0 \} = \{ j_1, \ldots, j_k \}, \text{ say.}$$

So $\tilde{x}$ uses columns $A_{\cdot j_1}, \ldots, A_{\cdot j_k}$ of A.

We use induction on k.

__Case 1__ $k = 0$:

Then $\tilde{x} = 0$, so $b = 0$. Thus $\tilde{x}$ is itself a BFS and a homog. soln,

so $\tilde{x} = 0 + 0$.

__Case 2__ $k \geq 1$:

Suppose the ~~theorem~~ result holds for every ~~point~~ feasible solution that uses a set of $k-1$ or less columns vectors of A.

If $\{A_{\cdot j_1}, \ldots, A_{\cdot j_k}\}$ is lin indep, $\tilde{x}$ is a BFS and the result holds.

So assume $\{A_{\cdot j_1}, \ldots, A_{\cdot j_k}\}$ is lin. dep.

Then $\exists\ \alpha_1, \ldots, \alpha_k$ not all zero such that

$$\alpha_1 A_{\cdot j_1} + \cdots + \alpha_k A_{\cdot j_k} = 0$$

~~Thus, $\tilde{x} + \theta\alpha$ is feasible for small enough θ.~~

Can assume WLOG some component of α is < 0.

$\tilde{x} + \theta\alpha$ feasible for small enough $\theta > 0$.

Let $\tilde{\theta} = \min_{j \in J} \left\{ \frac{\tilde{x}_j}{-\alpha_j} : \alpha_j < 0 \right\} > 0.$

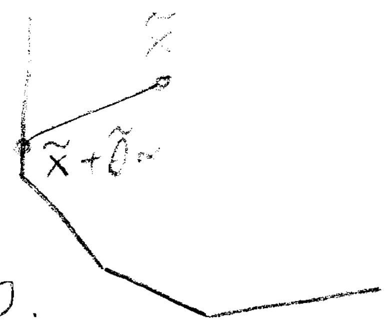
Then $\tilde{x} + \tilde{\theta}\alpha$ uses at most $k-1$ columns of A.

Two cases:

a) $\alpha \leq 0$:

Then, by induction hypothesis, $\tilde{x} + \tilde{\theta}\alpha = x' + y'$

where x' is a conv comb of bfs, y' is a homogeneous solution.



Thus $\tilde{x} = x' + (y' - \tilde{\theta}\alpha)$

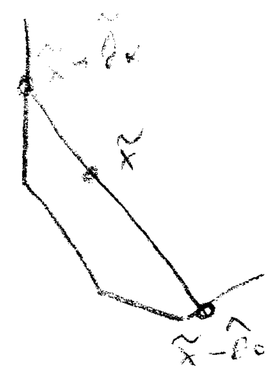
Now $y' \geq 0$, $\tilde{\theta} > 0$, $\alpha \leq 0$ so $y' - \tilde{\theta}\alpha \geq 0$.

Also, $A(y' - \tilde{\theta}\alpha) = Ay' - \theta A\alpha = 0$.

b) Some component of α is positive.

~~Then $\exists$ $\hat{\theta} > 0$ s.t.~~

~~Consider~~ $\tilde{x} - \theta\alpha$ feasible for small enough $\theta > 0$.



Let $\hat{\theta} = \min_{j \in J} \left\{ \frac{\tilde{x}_j}{\alpha_j} : \alpha_j > 0 \right\} > 0.$

Then $\tilde{x} - \hat{\theta}\alpha$ uses at most $k-1$ columns of A,

so $\tilde{x} - \hat{\theta}\alpha = x^2 + y^2$

where x^2 is a conv comb of bfs and y^2 is a homog soln.

We have

$$\tilde{x} = \frac{\hat{\theta}}{\hat{\theta}+\tilde{\theta}}\left(\tilde{x} + \tilde{\theta}\alpha\right) + \frac{\tilde{\theta}}{\hat{\theta}+\tilde{\theta}}\left(\tilde{x} \mp \hat{\theta}\alpha\right)$$

$$= \beta_1\left(\tilde{x} + \tilde{\theta}\alpha\right) + \beta_2\left(\tilde{x} + \hat{\theta}\alpha\right) \text{, say}$$

$$= \beta_1\left(x^1 + y^1\right) + \beta_2\left(x^2 + y^2\right)$$

$$= \cancel{\beta_1 + \beta_2}\ \underbrace{\beta_1 x^1 + \beta_2 x^2}_{\text{Conv. comb of bfs}} + \underbrace{\beta_1 y^1 + \beta_2 y^2}_{\text{Homogeneous soln.}}$$

//.

Let K be the feasible region of (P)

<u>Theorem</u> If (P) has a ~~bf~~ nondegenerate BFS, ~~it has a~~

~~dim (K~~ $\dim(K) = n-m$.

<u>Proof</u>: $\dim(K) \leq n-m$:

(P) has a nondegenerate bfs, so there are m lin. indep columns in A

Hence $\dim(K) \leq n-m$.

$\dim(K) \geq n-m$:

Let $\hat{x}$ be a nondegenerate bfs.

~~Assume~~

Rearrange cols if necessary so support of $\hat{x}$ is $\{1, \ldots, m\}$.

So have

$$\cancel{B^{-1}b} \quad x_i + \sum_{l=m+1}^{n} \bar{a}_i^l x_l = \bar{b}_i \qquad i=1,\ldots,m.$$

Lose [illegible] $\hat{x}$

We can move x_l away from zero by some small amount ε_l and still maintain feasibility, by adjusting $\hat{x}$ appropriately.

So define

$$\hat{\hat{x}}^l_i = \hat{x}_i - \varepsilon_l \bar{a}^l_i \qquad i=1,\ldots,m, \quad l=m+1,\ldots,n$$

$$\hat{\hat{x}}^l_j = \begin{cases} \varepsilon_l & \text{if } j=l \\ 0 & \text{o/w.} \end{cases}$$

Then the set of vectors $\hat{x} - \hat{\hat{x}}^l$ are lin indep

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$\dim(\ ) < n \cdot m$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \ \\ \ \end{pmatrix}$$

Only soln is $x_1 = \ldots = 0,\ x_3 = 0$

So $\dim(K) = 0$

$\dim(K) > n \cdot m$

$$\begin{pmatrix} 1 & 1 & 1 & \\ 1 & 0 & 0 & \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ \ \\ \ \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \ \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ 0 \\ \ \end{pmatrix}$$