Assume $\bar{c}_k < 0$:

Limits on the growth of x_k:

Need $x_B \geq 0$.

Recall $x_B = B^{-1}b - B^{-1}N x_N$

Only ~~$x_k \neq 0$~~ nonzero x_N is x_k, so

$$x_B = B^{-1}b - B^{-1}a^k x_k$$

ie

$$\begin{pmatrix} x_{B_1} \\ \vdots \\ x_{B_r} \\ \vdots \\ x_{B_m} \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_r \\ \vdots \\ \bar{b}_m \end{pmatrix} - \begin{pmatrix} \bar{a}_{1k} \\ \vdots \\ \bar{a}_{rk} \\ \vdots \\ \bar{a}_{mk} \end{pmatrix} x_k$$

$\bar{b} = B^{-1}b$

$\bar{a}_{\cdot k} = B^{-1}a^k =: \bar{a}^k$

$x_B^T = (x_{B_1}, \ldots, x_{B_r}, \ldots, x_{B_m})$

Two cases:

i) $\bar{a}_{ik} \leq 0 \quad \forall i \quad 1 \leq i \leq m$.

Now,
$$x_B = \cancel{B^{-1}}\,\bar{b} - \bar{a}^k x_k$$
$$\geq \bar{b} \geq 0.$$

So, can increase x_k arbitrarily.

So, objective function value

$$z = c_B B^{-1} b - \bar{c}_k x_k$$

is unbounded below.

ii) $\bar{a}_{ik} > 0$ for some i.

Increase x_k from 0 to $\Delta > 0$.

$x_i = \bar{b}_i - \bar{a}_i \Delta \geq 0$

So need $\Delta \leq \dfrac{\bar{b}_i}{\bar{a}_{ik}}$.

So take $\Delta = \min\limits_i \left\{ \dfrac{\bar{b}_i}{\bar{a}_{ik}} \;\middle|\; \bar{a}_{ik} > 0 \right\}$

Say minimum achieved by $\bar{a}_{rk}$, so $\Delta = \dfrac{\bar{b}_r}{\bar{a}_{rk}}$

So get $x_r = \bar{b}_r - \bar{a}_{rk} \cdot \dfrac{\bar{b}_r}{\bar{a}_{rk}} = 0$,

for $i \in B \setminus \{r\}$,

$$x_i = \bar{b}_i - \frac{\bar{b}_r}{\bar{a}_{rk}} \bar{a}_{ik}$$

$$\geq 0 \quad \cancel{\text{since } \frac{\bar{b}_i}{\bar{a}_{ik}} \geq \frac{\bar{b}_r}{\bar{a}_{rk}} \text{ if } \bar{a}_{ik} \geq 0}$$

Column a^k replaces column a^{B_r} in basis.

New BF solution:

$$\left.\begin{array}{ll} x_k = \bar{b}_r / \bar{a}_{rk} & \\ x_{B_i} = \bar{b}_i - \bar{a}_{ik} x_k = \bar{b}_i - \bar{a}_{ik} \dfrac{\bar{b}_r}{\bar{a}_{rk}} & i \neq r \end{array}\right\} \text{Basic variable}$$

$$\left.\begin{array}{ll} x_{B_r} = 0 & \\ x_j = 0 & j \in R \setminus \{k\} \end{array}\right\} \text{Nonbasic variables.}$$

min $x_1 + x_2 - 7x_3 - x_4$

s.t. $x_1 + x_2 + x_3 = 2$

$x_2 - 2x_3 + 2x_4 = 2$

$x_i \geq 0.$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$B^{-1}b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $B^{-1}N = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$

min 2 ~~[illegible]~~ $- 3x_3$ [illegible] x_4 $c_N^T - c_B^T B^{-1}N = [1 \ -1]$
$- [1 \ 0] = [-3 \ -1]$

s.t. x_1 $+ 3x_3 - 2x_4 = 0$ $c_B^T B^{-1}b = 2$

$x_2 - 2x_3 + 2x_4 = 2.$

$x \geq 0$

Pivot x_4 $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $N = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ $c_B^T = [1 \ -1]$
$c_N^T = [1 \ -2]$

$\begin{array}{cc} x_1 & x_4 \end{array}$ $\begin{array}{cc} x_2 & x_3 \end{array}$

x_4 [illegible] x_4

min $1 + \frac{3}{2}x_2 - 4x_3$

s.t. $x_1 + x_2 + x_3 = 2$

$\frac{1}{2}x_2 - x_3 + x_4 = 1$

Pivot x_3 [illegible]

min $-7 + 4x_1 + \frac{9}{2}x_2$ Optimal

s.t. $x_1 + x_2 + x_3 = 2$

$x_1 + \frac{3}{2}x_2 + x_4 = 3.$

$$\left[\begin{array}{cccc|c} 2 & 2 & \cancel{2} & 3 & \\ 1 & 2 & -1 & [illegible] & 2 \\ 0 & [illegible] & 1 & 0 & 1 \end{array}\right]$$

A[illegible]

(1, 3) [illegible]

[illegible]

$S = ($ [illegible]

[illegible] $= ($ [illegible] $)$ [illegible] $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$S =$ [illegible] $0 \; 2$ [illegible]

S [illegible] $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ [illegible]

Degeneracy

What does it mean if some component of x_B is ~~negative~~ zero?

Many bases correspond to this BFS.

Defn Let $\tilde{x}$ be a BFS of (P). $\tilde{x}$ is a nondegenerate BFS of (P) if exactly m of the components of $\tilde{x}$ are positive. If less than m of the components of $\tilde{x}$ are positive, $\tilde{x}$ is called a degenerate BFS.

Substitute $x_B = B^{-1}b - B^{-1}N x_N$ in objective function:

$$
\begin{aligned}
\min \quad & c_B^T B^{-1} b - c_B^T B^{-1} N x_N + c_N^T x_N \\
& x_B + B^{-1} N x_N = B^{-1} b \\
& x_B, x_N \geq 0.
\end{aligned}
$$

ie

$$
\begin{aligned}
\min \quad & (c_B^T B^{-1} b) + (c_N^T - c_B^T B^{-1} N) x_N \\
& x_B + B^{-1} N x_N = B^{-1} b \\
& x_B, x_N \geq 0.
\end{aligned}
$$

Defn The vector of reduced costs corresponding to the basis B is the vector

$$c_N^T - c_B^T B^{-1} N.$$

Theorem If the vector of reduced costs is positive, the point $x_B = B^{-1}b$, $x_N = 0$ is optimal, provided it is feasible.

Tableau:

Basis	X_N	x_k	$x_{B_1} \dots x_{B_r} \dots x_{B_m}$	RHS
$-z$	$c_N^T - c_B^T B^{-1} b$	$\bar{c}_k$	$0 \dots 0 \dots 0$	$-c_B B^{-1} b$
x_{B_1}	$B^{-1}N$		$1 \quad \dots \quad 0$	$B^{-1}b$
$\vdots$				
x_{B_r}		$\bar{a}_{rk}$	1	$\bar{b}_r$
$\vdots$				
x_{B_m}			$0 \quad \dots \quad 1$	

New solution is a bfs because $\bar{a}_{rk} > 0$, so after pivoting, get another identity matrix, so new B^{-1} exists, and new $(B^{-1}b)_r \geq 0$.

Revised [illegible]

Need: [illegible]

$\bar{b}$ [illegible]

$B^{-1}b$.

Never compute B^{-1}

Adjacency

Two extreme points $\tilde{x}$ and $\hat{x}$ of a convex polyhedron K are said to be adjacent iff every point $\bar{x}$ on the line segment joining them has the property that

if $\bar{x} = \alpha x^1 + (1-\alpha)x^2$, where $0<\alpha<1$ and $x^1 \in K$, $x^2 \in K$,

then both x^1 and x^2 must themselves be on the line segment joining $\tilde{x}$ and $\hat{x}$.

Simplex algorithm moves along edges of the polyhedron, from ~~vertex~~ a vertex to a neighbouring vertex which is strictly better.

Defn Let $\tilde{x} \in \mathbb{R}^n$, $\tilde{x} \neq 0$. The ray generated by $\tilde{x}$ is the set $\{x : x = \lambda \tilde{x}, \lambda \geq 0\}$

If $\hat{x} \in \mathbb{R}^n$, the set $\{x : x = \hat{x} + \theta \tilde{x}, \theta \geq 0\}$ is the half-line through $\hat{x}$ parallel to the ray generated by $\tilde{x}$.