Linear Programming

Consider the problem

$$\begin{aligned} \min \quad & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned} \qquad (P).$$

A is $m \times n$, ~~c, x~~ c, x are n vectors, b is an m-vector.

$n \geq m$.

Refer to (P) as standard form.

This formulation is general:

eg, consider

$$\begin{aligned} \min \quad & g^T x \\ & Hx \leq b \end{aligned}$$

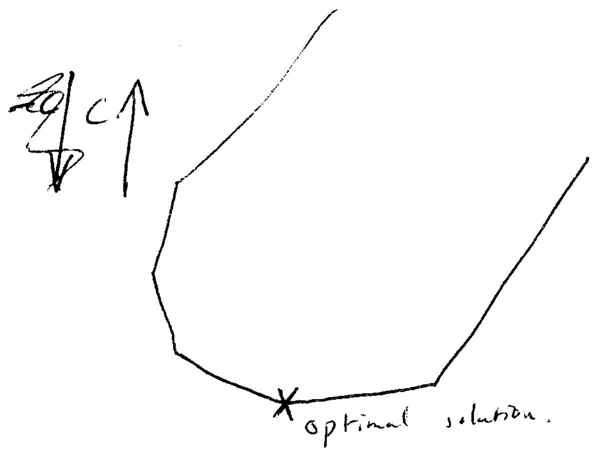
x unres.

Equiv to
$$\begin{aligned} \min \quad & g^T x \\ & Hx + Is = b \\ & s \geq 0 \end{aligned}$$

Equiv to
$$\begin{aligned} \min \quad & g^T (u - v) \\ & Hu - Hv + Is = b \\ & u, v, s \geq 0 \end{aligned}$$

$(x = u - v)$

Feasible region is a polyhedron, i.e. intersection of a finite number of closed half spaces.



Defns: For $C \subseteq \mathbb{R}^n$ convex, $x \in C$ is an extreme point of C if it can not be expressed as a convex combination of two other distinct points in C

~~Theorem~~ ~~Face: Let K be a pol the feasible region for~~

Face: Let K be a polyhedron. ~~Let H be a supporting hyperplane of K.~~ Then a face of K is either

i) K itself

ii) the empty set $\emptyset$

iii) the intersection of K and a supporting hyperplane H $K \cap H$.

Edge: a face of dimension 1

Facet: a face of dimension one less than the dimension of K.

Vertex: a face of dimension 0.

First define dimension:

S is a subspace of $\mathbb{R}^n$ if $x^1, x^2 \in S, \alpha, \beta \in S \Rightarrow \alpha x^1 + \beta x^2 \in S$

$\dim(S) =$ size of maximal linearly independent set of vectors in S

~~E~~ ~~A~~ $L \subseteq \mathbb{R}^n$ is an affine space if $x^1, x^2 \in L, \lambda \in \mathbb{R}$
$\Rightarrow \lambda x^1 + (1-\lambda)x^2 \in L$

If L is an affine space, $x^0 \in L$, then

$$L_S = \{x : x = y - x^0, y \in L\}$$ is a subspace.

Define $\dim(L) = \dim(L_S)$. (Exercise L_S does not ...)

~~Aff~~
Let $S \subseteq \mathbb{R}^n$, subset.

Affine hull of $S \equiv \text{aff}(S) = \{x : x = \sum_1^L \lambda_i x^i, \sum \lambda_i = 1, x^i \in S\}$

The $\dim(S) := \dim(\text{aff}(S))$.

Delay this:
Theorem The set of optimal feasible solutions of an LP is a face of the set of feasible solutions.

Proof Let K be the set of feasible solutions. If $K = \phi$, done.

If objective function is unbounded, set of optimal solns is empty.

Otherwise, let z be the optimal value.

Consider $H = \{x : c^T x = z\}$.

If x is feasible, $c^T x \geq z$, so H is a supporting hyperplane

Thus $K \cap H$ is a face of K, and it is the set of optimal solutions. //.

Defn ~~Any vector~~ Let K be the set of feasible solutions to (P). 59

If $\bar{x} \in K$, the set of column vectors of A that $\bar{x}$ uses is

$J_{\bar{x}} := \{ j : \bar{x}_j > 0 \} =$ support of $\bar{x}$. $\bar{x} \in K$ is a basic feasible solution (BFS) to (P) iff the set of columns of A that $\bar{x}$ uses is a linearly independent set. (Varied defn of for different formats).

Example:

x_1	x_2	x_3	x_4	x_5	b
1	0	-1	3	-1	1
0	1	1	4	2	4
0	0	0	-7	3	0

$x^T = (2,3,1,0,0)$ is <u>not</u> bfs since cols 1, 2, 3 not l.i.

$x^T = (1,4,0,0,0)$ <u>is</u> bfs.

<u>Theorem</u> Let K be the set of feasible solutions of (P). $x \in K$ is an extreme point of K iff it is a BFS.

<u>Proof</u> $\Rightarrow$ By contradiction.

Assume $\bar{x} \in K$ not bfs.

Then the columns of A used by $\bar{x}$ are linearly dependent, so

$\exists\, y \neq 0$ s.t. support of $y \subseteq$ support of $\bar{x}$, and $Ay = 0$

Let $\theta = \min\{\bar{x}_j : x_j > 0\}$. ~~Then~~

$\alpha = \max\{|y_j|\}$

Then $\bar{x} \pm \frac{\theta}{\alpha} y$ is feasible, so $\bar{x}$ not extreme

$\Leftarrow$ $\bar{x}$ is a BFS. Let J = support $(\bar{x})$.

If $\bar{x}$ not extreme. So $\bar{x} = \lambda \hat{x} + (1-\lambda)\tilde{x}$ for some $\hat{x}, \tilde{x} \neq \bar{x}$, $0<\lambda<1$.

$\bar{x}_j = 0 \quad \forall j \notin J$, so $\hat{x}_j = \tilde{x}_j = 0 \quad \forall j \notin J$.

Now $A\hat{x} = A\tilde{x} = b$, so $A(\hat{x} - \tilde{x}) = 0$

So columns in J are lin dep. $\#$ //.

<u>Theorem</u> (P) has a finite # of bfs's.

<u>Proof</u> Number of bfs is $\leq \binom{n}{m}$. //.

<u>Theorem</u> If (P) has a feasible solution, it has a bfs

<u>Proof</u> <u>Case 1</u>: If $b=0$, $x=0$ is a bfs.

<u>Case 2</u>: ~~If~~ Let $b \neq 0$. ~~In this case $x=0$ is not feasible~~

Let $\bar{x}$ be a feasible soln, $\bar{x}$ not bfs.

Let $J_{\bar{x}}$ be support of $\bar{x}$.

Set of columns corresponding to $J_{\bar{x}}$ is dependent.

So $\exists$ y with support$(y) \subseteq J_{\bar{x}}$, $Ay = 0$.

So $\exists\, \theta$ s.t. $\bar{x} + \theta y$ is feasible, and support $(\bar{x}+\theta y) \subsetneq J_{\bar{x}}$. //.

Theorem If (P) has an optimal solution, it has a bfs that is optimal.

Proof Similar to previous theorem //.

Theorem The set of optimal feasible solutions to (P) is a face of the set of feasible solutions.

Proof See p. 58 //.

So far, we've picked a point in $\mathbb{R}^n$, and examined its support to determine whether it is a BFS.

Consider choosing a subset of the columns of A, and finding a point corresponding to that set of columns.

So choose any m columns of A that are linearly independent.
WLOG, can assume they are the first m columns of A.
Denote these m columns by B, other columns by N, so

$$A \equiv [B \mid N].$$

So ~~constr~~ problem is

$$\begin{aligned} \min \quad & c_B^T x_B + c_N^T x_N \\ \text{s.t.} \quad & B x_B + N x_N = b \\ & x_B,\ x_N \geq 0. \end{aligned}$$

Since B invertible, constraints are equivalent to:

$$x_B + B^{-1} N x_N = B^{-1} b, \text{ so } x_B = B^{-1}b - B^{-1}N x_N$$

So one solution is

$$x_B = B^{-1} b, \quad x_N = 0.$$

This is the <u>basic solution</u> corresponding to B.

It is a BFS if $B\, x_B \geq 0$.

Objective function value $= c_B^T x_B = c_B^T B^{-1} b$

Simplex Algorithm:

$$\begin{array}{ll} \min & c^T x \\ & A x = b \\ & x \geq 0 \end{array} \qquad (P)$$

$$\begin{array}{ll} \min & c_B^T x_B + c_N^T x_N \\ & B x_B + N x_N = b \\ & x_B, x_N \geq 0 \end{array}$$

$$\begin{array}{ll} \min & c_B^T x_B + c_N^T x_N \\ & x_B + B^{-1} N x_N = B^{-1} b \\ & x_B, x_N \geq 0 \end{array}$$

~~Assume $B^{-1}b \geq 0$.~~

$$\begin{array}{ll} \min & c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N \\ & x_B + B^{-1} N x_N = B^{-1} b \\ & x_B, x_N \geq 0. \end{array}$$

Assume $B^{-1} b \geq 0$.

Notation: a^k : kth column of A. R = (column) index set of N

$$\bar{c}_k = c_k - c_B^T B^{-1} a_k \qquad k \in R$$

reduced cost.

Objective function value:

$$z = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N$$

$$= c_B^T B^{-1} b + \sum_{k \in R} \bar{c}_k x_k$$

Theorem If all $\bar{c}_k$ are nonnegative, the point $x_B = B^{-1}b$, $x_N = 0$ is optimal.

Proof For $x_B = B^{-1}b$, $x_N = 0$, we have $z = c_B^T B^{-1} b$

For any other feasible x, we have

$$z = c_B B^{-1} b + \sum_{k \in R} \bar{c}_k x_k$$
$$\geq c_B B^{-1} b$$

//.

Theorem If some $\bar{c}_k$ is negative, the point $x_B = B^{-1}b$, $x_N = 0$ is not optimal, provided $B^{-1}b > 0$.

Proof ~~We can increase~~
Keep all $j \in R \setminus \{k\}$ at zero, increase x_k from zero.

We can balance the equations using x_B.

New obj fn value is

$$z = c_B B^{-1} b + \sum_{j \in R} \bar{c}_j x_j$$
$$= c_B B^{-1} b + \bar{c}_k x_k < c_B B^{-1} b.$$

//.