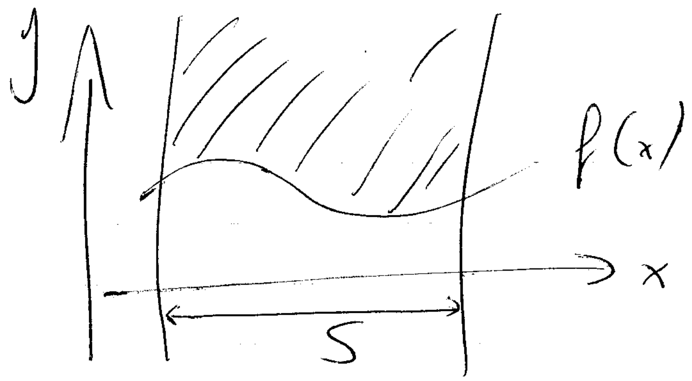
CONVEX FUNCTIONS

Defn Given a convex set $C \subseteq \mathbb{R}^n$,
$f : C \to \mathbb{R}$ is called <u>convex</u> on C if

$$\left.\begin{matrix} x \in C \\ y \in C \\ \lambda \in (0,1) \end{matrix}\right\} \Rightarrow f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$$

Defn Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$

The <u>epigraph</u> of $f = \{ (x,y) \mid x \in S, y \in \mathbb{R}, f(x) \le y \}$.



Thm ~~f is convex $\Leftrightarrow$ epi f is convex.~~

Let C be a nonempty convex set in $\mathbb{R}^n$.

Let $f : C \to \mathbb{R}^*$. Then

f is convex $\Leftrightarrow$ epi f is convex.

Proof $\Rightarrow$: Assume f convex

Let (x^1, y^1) and (x^2, y^2) be in epi f.

Consider $\lambda(x^1, y^1) + (1-\lambda)(x^2, y^2) = (\tilde{x}, \tilde{y})$, $0 < \lambda < 1$

Then
$$\begin{aligned}
\tilde{x} &= \lambda x^1 + (1-\lambda) x^2 \\
\tilde{y} &= \lambda y^1 + (1-\lambda) y^2 \\
&\geq \lambda f(x^1) + (1-\lambda) f(x^2) \\
&\geq f(\lambda x^1 + (1-\lambda) x^2) = f(\tilde{x}).
\end{aligned}$$

So $\tilde{y} \geq f(\tilde{x})$.

$\Leftarrow$: Assume epi f convex:

Let $x^1, x^2 \in C$, $\lambda \in (0,1)$.

~~Then~~ $\cancel{f(\lambda x^1 + (1-\lambda) x^2)}$

$\cancel{\lambda f(x^1) + (1-\lambda) f(x^2) \in \text{epi } f}$ ~~since~~

& $(x^1, f(x^1)), (x^2, f(x^2)) \in$ epi f

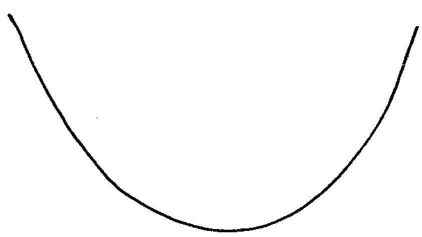
$\therefore \lambda(x^1, f(x^1)) + (1-\lambda)(x^2, f(x^2)) \in$ epi f

$\therefore f(\lambda x^1 + (1-\lambda) x^2) \leq \lambda f(x^1) + (1-\lambda) f(x^2)$. //

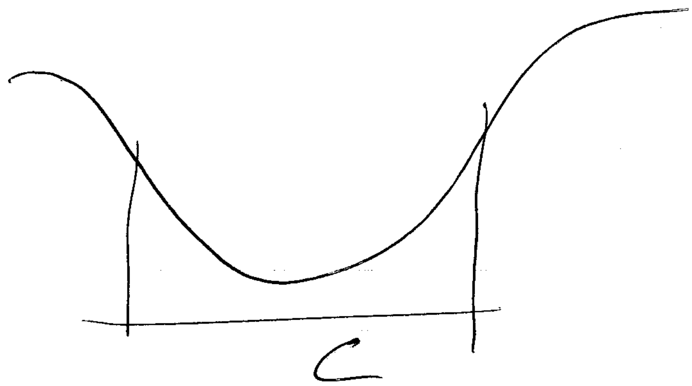
Defn $C_\alpha = \{x \in C \mid f(x) \le \alpha\}$ is called the level set associated with f.

Lemma f convex $\Rightarrow$ all its level sets are convex (converse not true).

Examples i)

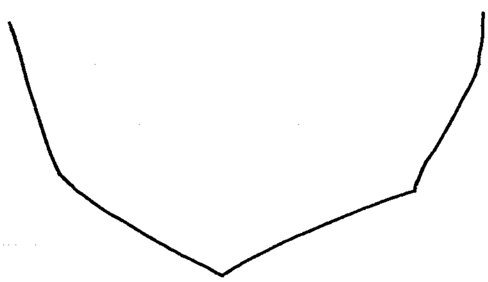


ii)



iii) $c^T x + \frac{1}{2} x^T B x$ is convex ~~provided~~ iff B is positive semi definite.

iv)



<u>Fact</u> f convex $\Rightarrow$ f is continuous on int C

(Proof: Involved, see text).

<u>THM</u>. Let f be a convex fcn defined on a convex set $C \subseteq \mathbb{R}^n$. Let $\bar{x} \in C$ and let d be any <u>feasible direction</u>, ie $\bar{x} + \lambda d \in C$ for some $\lambda > 0$. Then the <u>difference quotient</u> $\dfrac{f(\bar{x}+\lambda d) - f(\bar{x})}{\lambda}$

is a nondecreasing function of $\lambda > 0$.

<u>Remark</u> As a consequence, the directional derivative

$$D_d f(\bar{x}) = \lim_{\substack{\lambda \to 0 \\ \lambda > 0}} \frac{f(\bar{x}+\lambda d) - f(\bar{x})}{\lambda}$$

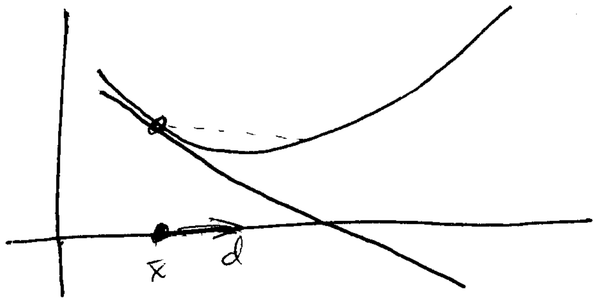
exists (provided we allow ~~$\infty, -\infty$~~ $\pm\infty$).

Note that

$$D_d f(\bar{x}) = \nabla f(\bar{x})^T d ,$$

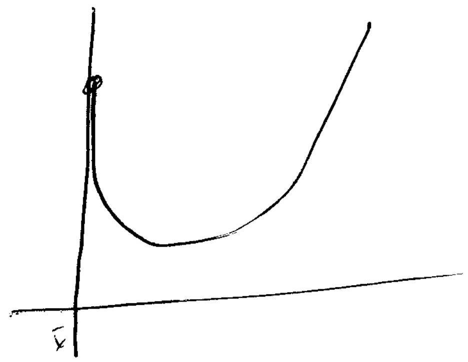
provided $\nabla f(\bar{x})$ exists.

Picture



$$\frac{f(\bar{x}+\lambda d)-f(\bar{x})}{\lambda} = \text{slope}$$

Could get $-\infty$:



Proof Let $0 < \lambda_1 < \lambda_2$.

We have

$$f(\bar{x}+\lambda_1 d) = f\left(\frac{\lambda_1}{\lambda_2}(\bar{x}+\lambda_2 d) + \left(1-\frac{\lambda_1}{\lambda_2}\right)\bar{x}\right)$$

$$\leq \frac{\lambda_1}{\lambda_2} f(\bar{x}+\lambda_2 d) + \left(1-\frac{\lambda_1}{\lambda_2}\right) f(\bar{x})$$

Thus, $$\frac{f(\bar{x}+\lambda_1 d)-f(\bar{x})}{\lambda_1} \leq \frac{f(\bar{x}+\lambda_2 d)-f(\bar{x})}{\lambda_2}.$$

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