Strict Separation

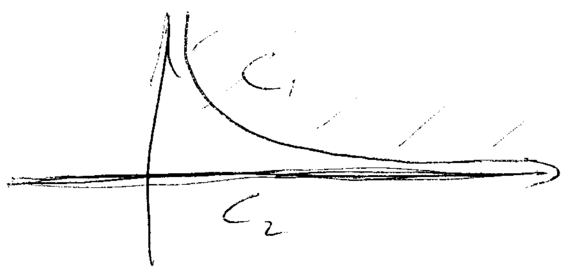
Two ~~set~~ convex sets C_1, C_2 disjoint.

Can we find p, a s.t. $\sup_{x \in C_1} \{p^T x\} < a < \inf_{y \in C_2} \{p^T y\}$ for all $x \in C_1$, $y \in C_2$?

C_1 closed, C_2 open:

Not possible.

C_1, C_2 closed:



C_1 closed, C_2 compact (closed & bounded):

Yes: Consider $y \in C_2$.

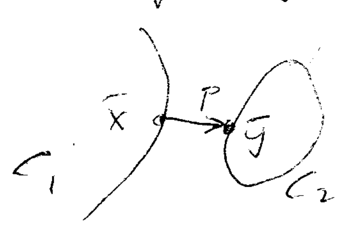
Define $f(y) = \min_{x \in C_1} \|y - x\| > 0 \quad \forall y$.

~~Define~~ Since C_2 compact, $\min f(y)$ is attained.

So let $\bar{y}$ attain min, with $\bar{x} \in C_1$ satisfying

$$\|\bar{y} - \bar{x}\| = f(\bar{y}) = \min_{y \in C_2} f(y)$$

Define $p = \bar{y} - \bar{x}$. $\quad a = p^T \bar{x} + \frac{1}{2} p^T p$



$\bar{x}$ is closest point in C_1 to $\bar{y}$

$\bar{y}$ — — C_2 — $\bar{x}$.

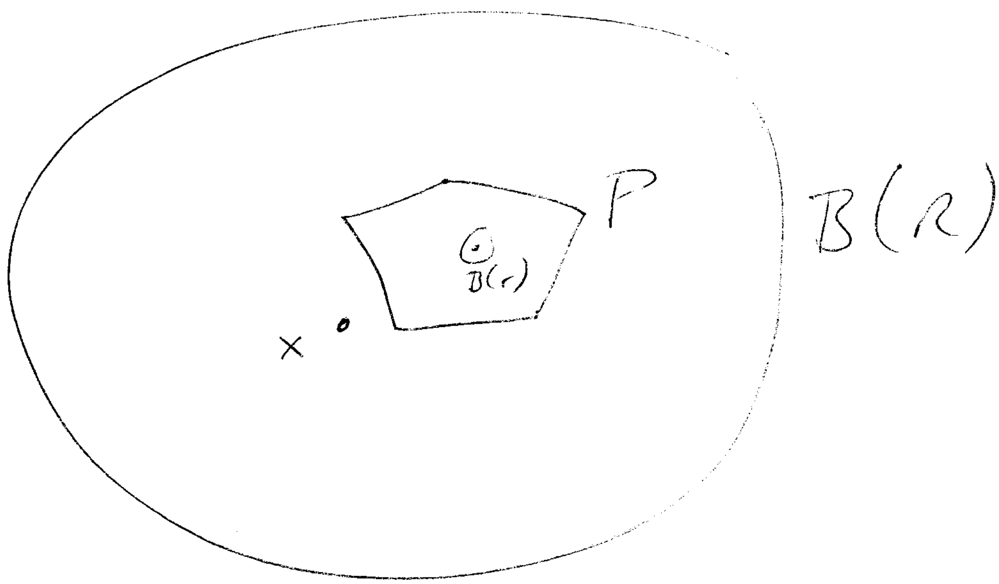
Ellipsoid Algorithm.

Given: set of linear constraints $Ax \le b = P$, a closed convex set.

Assume: polyhedron has nonempty interior, ~~for~~ can fit a ball of radius r inside the polyhedron.

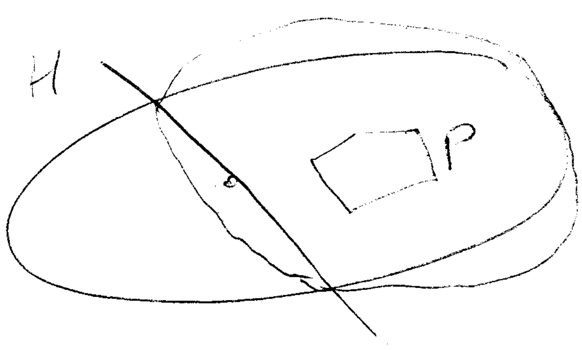
Want to find a feasible point.

Assume: polyhedron is contained in a ball of radius R.



Want to shrink containing ball until its center is in P.

Do this by separating center x from polyhedron P, ~~gen~~ generating a new ellipsoid which contains the correct half-ellipsoid



$$\frac{\text{vol (new ellipsoid)}}{\text{vol (old ellipsoid)}} < e^{\frac{-1}{2(n+1)}} \quad n = \text{dimension of } x.$$

So after k iterations,

$$\frac{\text{vol}(E^k)}{\text{vol}(E^0)} \leq e^{-\frac{k}{2(n+1)}} \qquad (*)$$

Now $\text{vol}(E^0) \equiv$ ~~πR^2~~ $\pi(2R)^n$.

To get a bound on the number of iterations needed, $P \subseteq E^k$; so always have $\text{vol}(E^k) \geq \pi(2r)^n$ if center $\notin P$.

~~So need no more than~~

Rearranging $(*)$ gives

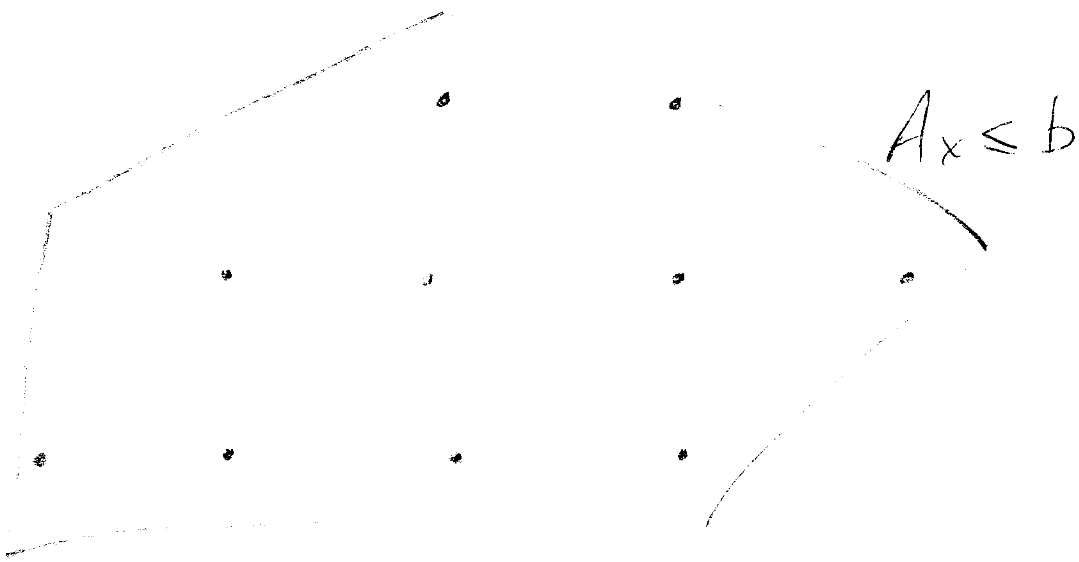
$$2(n+1)\ln\left(\frac{\text{vol}(E_0)}{\text{vol}(E_k)}\right) \geq k$$

So ~~the~~ algorithm ~~converges~~ stops ~~by~~ for some $k \leq 2(n+1)\, n \ln\left(\frac{R}{r}\right)$.

Cutting Plane Methods for Integer Programming

Consider integer program

$$\begin{aligned} \min \quad & c^T x \\ & Ax \le b \\ & x \text{ integer.} \end{aligned} \qquad (IP)$$



Let S = set of feasible integer points.

Then $\mathrm{conv}(S)$ is a closed convex set, a polyhedron.

Extreme points of $\mathrm{conv}(S)$ are integer points,

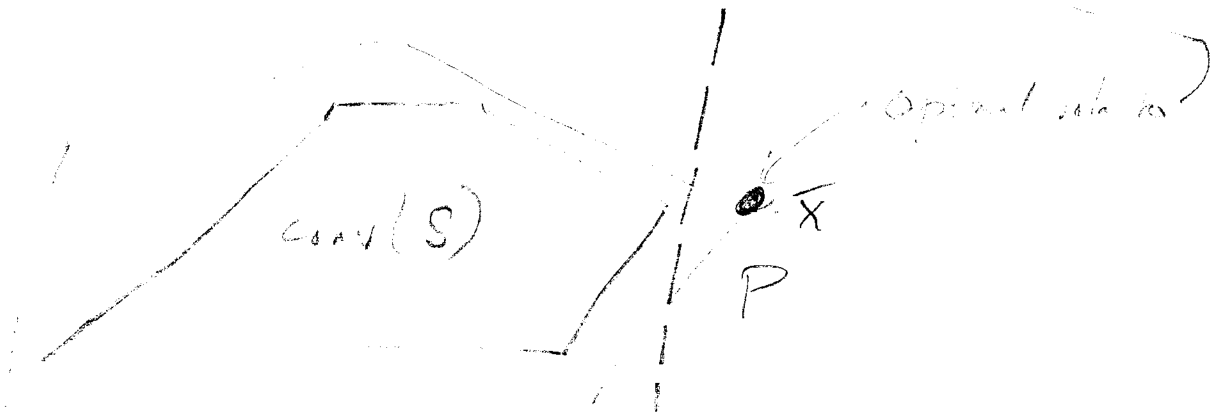
so solving $\quad \min c^T x$ s.t. $x \in \mathrm{conv}(S)$

gives optimal solution to IP.

Problem: hard to get a description of $\mathrm{conv}(S)$.

So drop integrality requirement.

Solve $\min c^T x$
$Ax \leq b$



Can separate $\bar{x}$ from conv(S).

eg $p^T x \leq \alpha$ for all x in conv(S).

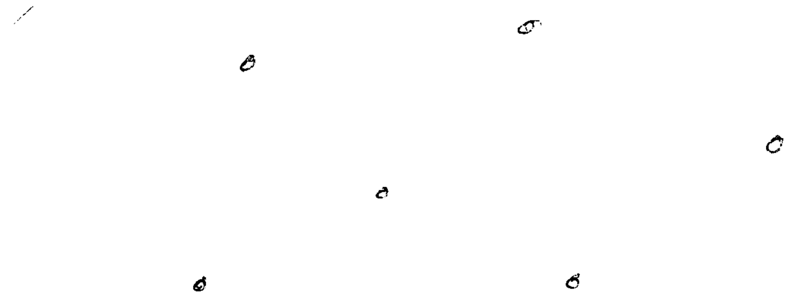
So now solve relaxation

$$\begin{aligned} \min\ & c^T x \\ & Ax \leq b \\ & p^T x \leq \alpha, \end{aligned}$$

and repeat.

Do: Eg. Knapsack, because of the applicability [illegible]

Eg: Travelling Salesman Problem: Collection of cities. Want to start from home base, and visit each city exactly once.



Let edge e_i have length ~~e_i~~ c_i

Let $x_i = \begin{cases} 1 & \text{if use edge } e_i \\ 0 & \text{o/w} \end{cases}$

$$\min \quad \sum c_i x_i$$

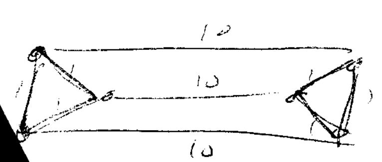
s.t. $x_{\cancel{i}}$ represent a tour.

$$\min \quad \sum c_i x_i$$

$$\text{s.t.} \quad \sum_{\substack{\text{edges } e_i \\ \text{adjacent to vertex } j}} x_{e_i} = 2 \quad \text{for all vertices } j$$

$$x_{e_i} \geq 0 \quad \text{for all edges } e_i.$$

x integral

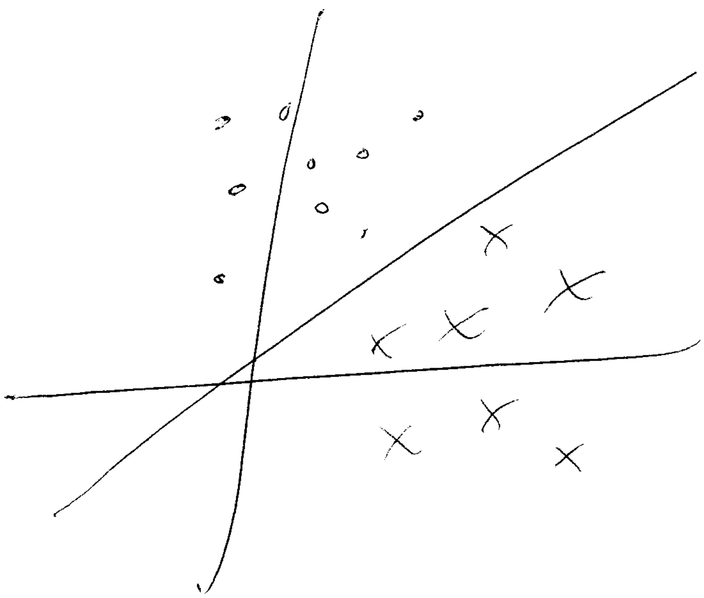


Also need subtour elimination constraints.

Clustering.

Have points in $\mathbb{R}^n$. Some possess characteristic K, some don't.

Want to separate the sets.



May not be able to separate. ~~To generalize to cases with~~ Extend to handle such cases.