Separation Theorem

Important Fact

Given C a nonempty closed convex set in $\mathbb{R}^n$.

Given $y \notin C$

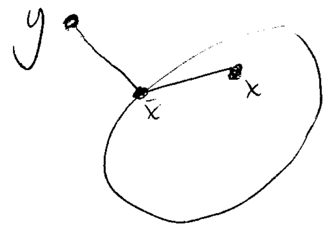
Then $\exists\, \bar{x} \in C$ such that

i) $\min_{x \in C} \|y - x\| = \|y - \bar{x}\|$

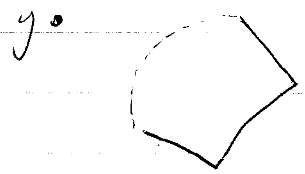
ii) $\bar{x}$ is unique

iii) ~~$\bar{x}$ is the~~ $(x - \bar{x})^T (y - \bar{x}) \leq 0 \quad \forall x \in C$.

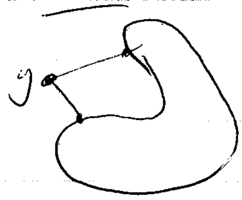
Proof See text. //



Assumption closed is important:



Assumption convex is important



Defn A hyperplane $H(p,a) = \{x \mid p^T x = a\}$ is said to <u>separate</u> two nonempty sets A and B in $\mathbb{R}^n$ if

$$x \in A \Rightarrow p^T x \geq a$$

$$x \in B \Rightarrow p^T x \leq a$$

(strictly separate if strict ineqs.)

Theorem Given C a nonempty closed convex set in $\mathbb{R}^n$, $y \notin C$.

Then $\exists\, p \neq 0$ and a scalar a such that

$$p^T x \leq a < p^T y \qquad \forall x \in C.$$

Proof Let $\bar{x}$ be the unique min pt for $\min_{x \in C} \|y - x\|$

We know $(x-\bar{x})^T(y-\bar{x}) \leq 0 \quad \forall x \in C.$

$$\|y-\bar{x}\|^2 = (y-\bar{x})^T(y-\bar{x}) = y^T(y-\bar{x}) - \bar{x}^T(y-\bar{x})$$

$\leq \bar{x}^T(y-\bar{x})$

$$\leq y^T(y-\bar{x}) - x^T(y-\bar{x})$$

$$= (y-x)^T(y-\bar{x}) \qquad \forall x \in C.$$

Letting $p = y - \bar{x}$, we obtain $p \neq 0$ and

$$p^T(y-x) \geq \|p\|^2 \qquad \forall x \in C$$

$$\Rightarrow p^T y \geq p^T x + \|p\|^2 \qquad \forall x \in C$$

Let $a = \sup\{p^T x \mid x \in C\} \Rightarrow p^T y \geq a + \|p\|^2 > a$

$\therefore\ p^T x \leq a \quad \forall x \in C$ and $p^T y > a$ //

<u>Corollary</u>

~~C~~ If C is a closed convex set, it is the intersection of all the halfspaces containing it.

<u>Proof</u> ~~C~~ $C \subseteq$ intersection of halfspaces containing it: Obvious.

Conversely:
Assume $\exists\, y \notin C$, but y is in all halfspaces containing ~~C~~ C

By theorem, $\exists$ halfspace H with $C \subseteq H$, $y \notin H$ #

//.

THM Let C be a nonempty convex set in $\mathbb{R}^n$. Let $y \in \partial C$.

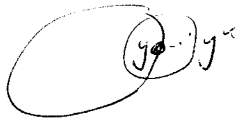
Then $\exists\, p \neq 0$ such that $p^T x \leq p^T y \quad \forall x \in \mathrm{cl}\, C$.

Pf $y \in \partial C \Rightarrow \exists \{y^k\}$ not in $\mathrm{cl}\, C$ s.t. $y^k \to y$

Diag: this.

(Need convexity for this: eg

Holds because:



)

By the previous theorem,

$$\exists\, p^k \neq 0 \quad \text{s.t.} \quad (p^k)^T y^k > (p^k)^T x \quad \forall x \in \mathrm{cl}\, C$$

(cl C is convex).

WLOG it may be assumed that $\|p^k\| = 1$

Since $\{p^k\}$ is bounded, it has a convergent subsequence,

$\{p^k\}_{k \in K}$, ~~with~~ with $p := \lim_{k \in K} p^k$.

Since $(p^k)^T y^k > (p^k)^T x \quad \forall x \in \mathrm{cl}\, C$,

we have $$\lim_{k \in K} \left[(p^k)^T y^k \right] \geq \lim_{k \in K} (p^k)^T x \quad \forall x \in \mathrm{cl}\, C$$

$$\Rightarrow \quad p^T y \geq p^T x \quad \forall x \in \mathrm{cl}\, C \qquad //.$$

<u>Corollary</u> (of above two theorems)

Let C be a nonempty convex set in $\mathbb{R}^n$

If $y \notin C$ then $\exists$ $p \neq 0$ such that $p^T x \leq p^T y$ for each $x \in \text{cl}\, C$

<u>Proof</u> If $y \notin \text{cl}\, C$, follows from first theorem

If $y \in \text{cl}\, C$, but not C, then $y \in \partial C$, so result follows from second theorem //.

Theorem Let C_1 and C_2 be nonempty, disjoint convex sets in $\mathbb{R}^n$. ~~If~~ Then $\exists\, p \neq 0$ and a scalar a such that

$$\sup_{x \in C_1} \{p^T x\} \leq a \quad \forall x \in C_1$$

$$\inf_{x \in C_2} \{p^T x\} \geq a \quad \forall x \in C_2$$

Proof $C := C_1 - C_2$ is convex and $0 \notin C_1 - C_2$.

(see p.9 for defn of $C_1 + C_2$)

$\therefore \exists\, p \neq 0$ such that

$$p^T x \leq p^T 0 = 0 \quad \forall x \in C.$$

Thus $p^T x^1 \leq p^T x^2 \quad \forall x^1 \in C_1,\ x^2 \in C^2$

$$\Rightarrow \quad \alpha = \sup \{p^T x \mid x \in C^1\} \leq \inf \{p^T x \mid x \in C^2\} = \beta$$

So $p^T x \leq \alpha \quad \forall x \in C_1$

$p^T x \geq \alpha \quad \forall x \in C_2.$ //

Both closed, one bounded $\Rightarrow$ strict separation.

Illustrate need bounded

Separating hyperplanes used in

ellipsoid algo

cutting plane methods

Fill out lecture with these.