Defn Given $S \subseteq \mathbb{R}^n$, the convex hull of S, $\text{conv}(S)$, is the intersection of all convex sets containing S. (ie, the smallest convex set that contains S).

Lemma $\text{conv}(S)$ is the set of convex combinations of points in S.

Proof: Let M be the set of all convex combinations of points in S. Finite.

First, show $\text{conv}(S) \subseteq M$.

It suffices to show that M is convex and $S \subseteq M$.

$S \subseteq M$: obvious.

M convex: Pick two points, x and y in M

So $x = \sum_1^k \lambda_i a^i$, $\sum_1^k \lambda_i = 1$, $\lambda_i \geq 0$, and $a^i \in S$

$y = \sum_1^l \mu_j b^j$, $\sum_1^l \mu_j = 1$, $\mu_j \geq 0$, and $b^j \in S$

Thus for $0 \leq \delta \leq 1$ we have

$$\delta x + (1-\delta) y = \sum_1^k (\delta \lambda_i) a^i + \sum_1^l ((1-\delta)\mu_j) b^j$$

~~Claim~~
Coefficients $\delta\lambda_i$ and $(1-\delta)\mu_j$ are nonnegative.

Also, $\sum_1^k \delta\lambda_i + \sum_1^l (1-\delta)\mu_j$

$$= \delta + (1-\delta) = 1.$$

Thus, $\delta x + (1-\delta) y \in M$

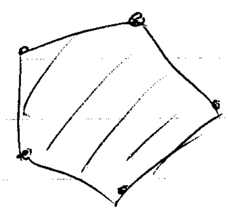
So M is convex

Second: Show $M \subseteq \mathrm{conv}(S)$

$\mathrm{conv}(S)$ is convex and contains S. So every convex combination of points in S is in $\mathrm{conv}(S)$ by our previous theorem. $\therefore M \subseteq \mathrm{conv}(S)$

Thus $M = \mathrm{conv}(S)$ //

<u>Definition</u>: The convex hull of a finite number of points in $\mathbb{R}^n$ is called a polytope:

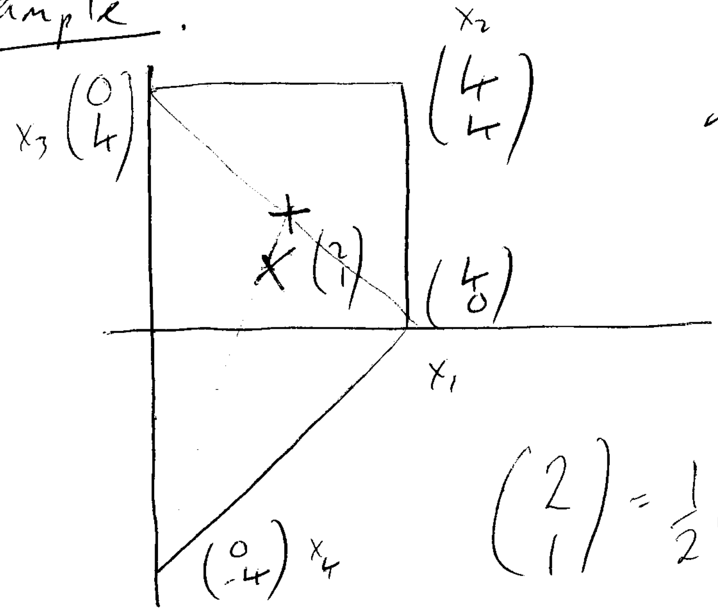


<u>Definition</u>: If $x^2 - x^1, \ldots, x^k - x^1$ are linearly independent then ~~the~~ $\mathrm{conv}\{x^1, \ldots, x^k\}$ is called a simplex:

eg in $\mathbb{R}^2$: 

In $\mathbb{R}^3$: tetrahedron.

Example.



$$\frac{1}{4}\begin{pmatrix}4\\0\end{pmatrix}+\frac{1}{4}\begin{pmatrix}4\\4\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0\\4\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0\\-4\end{pmatrix}$$
$$=\begin{pmatrix}2\\1\end{pmatrix}.$$

$$\begin{pmatrix}2\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4\\0\end{pmatrix}+\frac{3}{8}\begin{pmatrix}0\\4\end{pmatrix}+\frac{1}{8}\begin{pmatrix}0\\-4\end{pmatrix}$$

After Carathéodory's Theorem:

$$x_2 - x_1 = \begin{pmatrix}0\\4\end{pmatrix} \quad x_3 - x_1 = \begin{pmatrix}-4\\4\end{pmatrix} \quad x_4 - x_1 = \begin{pmatrix}-4\\-4\end{pmatrix}$$

$$\mu_2 = 2,\ \mu_3 = -1,\ \mu_4 = 1 \Rightarrow \mu_1 = -2.$$

Since all λ_i's $= \frac{1}{4}$,
~~So~~ ratio achieved by largest μ_i, ie μ_2

Get $\underset{\cancel{\lambda_1}}{} \quad \alpha = \dfrac{\lambda_2}{\mu_2} = \dfrac{1}{8}$

So $\lambda_1 - \alpha\mu_1 = \frac{1}{2}$, $\lambda_2 - \alpha\mu_2 = 0$, $\lambda_3 - \alpha\mu_3 = \frac{3}{8}$, $\lambda_4 - \alpha\mu_4 = \frac{1}{8}$

(If time: consider $\mu_2 = -2, \mu_3 = 1, \mu_4 = -1$).

Carathéodory's Thm

Let S be an arbitrary set in ~~E_n~~ $\mathbb{R}^n$.

~~If $x \in \text{conv}(S)$, then $x \in \text{conv}(\{x_1, \ldots, x_{n+1}\})$, where $x_j \in S$ for $j = 1, \ldots, n+1$.~~

x can be expressed as a convex combination of $n+1$ or fewer points in S.

Proof Suppose $x = \sum_1^k \lambda_j x^j$, $k \geq n+2$,

with $\lambda_j > 0$, $\sum \lambda_j = 1$, $x^j \in S$.

Will show x can be expressed as a conv. comb of $k-1$ points in S.

Consider

$x_2 - x_1, \ldots, x_k - x_1$ — $k-1$ vectors

Lin. dep. since $k-1 > n$.

$\therefore \exists\, \mu_j$ not all zero such that

$$\mu_2(x_2 - x_1) + \cdots + \mu_k(x_k - x_1) = 0$$

Set $\mu_1 = -\sum_2^k \mu_j$

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Then $\sum_1^k \mu_j = 0$ – so at least one $\mu_j > 0$.

and
$$\sum_1^k \mu_j x_j = \cancel{\Sigma_j}\, \mu_1 x_1 + \sum_2^k \mu_j x_j$$
$$= -x_1 \sum_2^k \mu_j + \sum_2^k \mu_j x_j$$
$$= \sum_2^k \mu_j (x_j - x_1) = 0.$$

~~Consider the point~~

Thus the point
$$\sum_1^k (\lambda_j - \alpha \mu_j) x_j = \sum_1^k \lambda_j x_j = x \quad \text{for all } \alpha.$$

So choose α so one of the coeffs $\lambda_j - \alpha\mu_j$ is zero, and others are nonnegative.

In particular, set $\alpha = \min\left\{ \frac{\lambda_j}{\mu_j} : \mu_j > 0 \right\}$.

Also, since $\Sigma \mu_j = 0$, we have $\sum_1^k (\lambda_j - \alpha\mu_j) = 1$.

//

ALGORITHMIC Proof.

Topological Properties

Defn Given $S \subseteq \mathbb{R}^n$,

the closure of S is

$$\mathrm{cl}\, S := \{x \mid N_\varepsilon(x) \cap S \neq \emptyset \text{ for every } \varepsilon > 0\}$$

(every nbhd contains at least one pt of S)

where $N_\varepsilon(x) := \{y \mid \|y - x\| < \varepsilon\}$

S is called closed if $\mathrm{cl}\, S = S$.

eg $\{x \geq 0 \mid x_1 + x_2 < 1\}$:

Interior of S is

$$\mathrm{int}\, S := \{x \mid N_\varepsilon(x) \subseteq S \text{ for some } \varepsilon > 0\}$$

(some nbhd is contained in S).

S is called open if $\mathrm{int}\, S = S$.

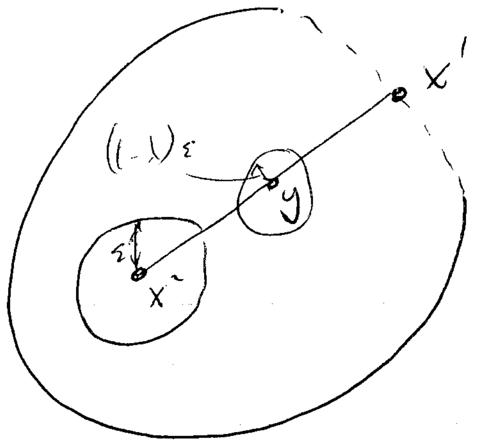
Boundary of S is

$$\partial S := \{x \mid \text{for every } \varepsilon > 0,\ N_\varepsilon(x) \text{ contains a point in } S \text{ and a point not in } S\}$$

Theorem $C \subseteq \mathbb{R}^n$ is convex with $\text{int } C \neq \emptyset$.

$$\left.\begin{array}{l} x^1 \in \text{cl } C \\ x^2 \in \text{int } C \end{array}\right\} \Rightarrow \lambda x^1 + (1-\lambda)x^2 \in \text{int } C \text{ for each } \lambda, \; 0 < \lambda < 1.$$

Proof.



Pick $\varepsilon > 0$ so $N_\varepsilon(x^2) \subseteq C$.

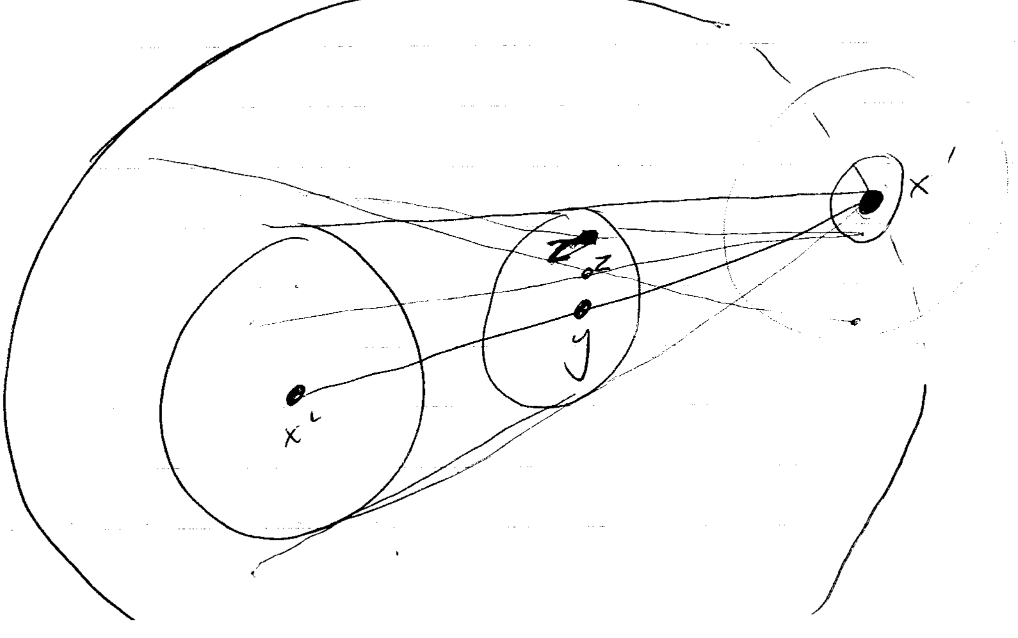
Let $y = \lambda x^1 + (1-\lambda)x^2$ for some $\lambda \in (0,1)$.

Want to show $y \in \text{int } C$.

Claim $N_{(1-\lambda)\varepsilon}(y) \subseteq C$.

Pick $z \in N_{(1-\lambda)\varepsilon}(y)$.

Show $z \in C$.



Pick $z' \in N_{\left[\frac{(1-\lambda)\varepsilon - \|z-y\|}{\lambda}\right]}(x') \cap S$

~~Choose~~ Find $z^2 \in N_\varepsilon(x^2)$ so that

$\cancel{z = \lambda z' + (1-\lambda)}$

$z = \lambda z' + (1-\lambda) z^2$ ie $z^2 = \dfrac{z - \lambda z'}{1-\lambda}$.

<u>Claim</u> $z^2 \in N_\varepsilon(x^2)$:

$$\|z^2 - x^2\| = \left\| \frac{z - \lambda z'}{1-\lambda} - x^2 \right\|$$

$$= \left\| \frac{z - \lambda z'}{1-\lambda} - \frac{y - \lambda x'}{1-\lambda} \right\|. \quad \text{Subst for } x_2$$

$$= \frac{1}{1-\lambda} \| z - y - \lambda(x' - z')\| \quad \text{Take out } \tfrac{1}{\cdot} \text{ factor}$$

$$\le \frac{1}{1-\lambda} \left[\|z-y\| + \lambda \|x' - z'\| \right] \quad \Delta\text{-ineq}$$

$$< \frac{1}{1-\lambda} \left[\|z-y\| + (1-\lambda)\varepsilon - \|z-y\| \right] \quad z' \in N_\cdot(x$$

$$= \varepsilon \quad //.$$

Consequences.

Corollary If S is convex then $\operatorname{int} S$ is convex.

Pf Let $x^1, x^2 \in \operatorname{int} S$.

Let $y = \lambda x^1 + (1-\lambda) x^2$.

~~Now $x^1 \in \varepsilon$~~

~~Now $\operatorname{int} S \subseteq S \subseteq \operatorname{cl} S$, so~~

Now $\operatorname{int} S \subseteq \operatorname{cl}(\operatorname{int} S)$, so $x^1 \in \operatorname{cl}(\operatorname{int} S)$.

Therefore $y \in \operatorname{int} S$, by previous theorem. //

Corollary If S is convex and $\operatorname{int} S \neq \emptyset$,

i) $\operatorname{cl} S$ is convex

ii) $\operatorname{cl}(\operatorname{int} S) = \operatorname{cl} S$

iii) $\operatorname{int}(\operatorname{cl} S) = \operatorname{int} S$.

(are relative interiors, [illegible] $\operatorname{int} S \neq \emptyset$ [illegible] $\operatorname{relint}(S) \neq \emptyset$ provided $S \neq \emptyset$

(Nonconvex counter exs to

ii) $S = [0,1] \cup$ ~~[illegible]~~ $\{2\}$

Then $\operatorname{cl}(S) = S$, $\operatorname{int} S = (0,1)$, $\operatorname{cl}(\operatorname{int}(S)) = [0,1]$

iii) $S = [-1,0) \cup (0,1]$

Then $\operatorname{cl}(S) = [-1,1]$, $\operatorname{int} \operatorname{cl} S = (-1,1)$, $\operatorname{int} S = (-1,0) \cup (0,1)$

(int $S = \emptyset$ counter examples:

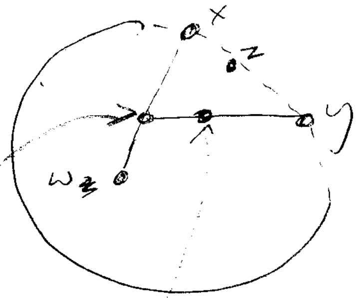
ii) $S = \{1\} \subseteq \mathbb{R}$)

([illegible]

Proof i) Pick $x \in cl\, S$, $y \in cl\, S$

Let $z = \lambda x + (1-\lambda) y$

Want to show $z \in cl\, S$.



Pick $w \in int\, S$.

By theorem, $\mu x + (1-\mu) w \in int\, S$

So by theorem,

$$\lambda(\mu x + (1-\mu) w) + (1-\lambda) y \in int\, S$$

True $\forall \mu \in (0,1)$.

Therefore, taking limit as $\mu \to 1$ gives a point in $cl\, S$

ie $\lambda x + (1-\lambda) y \in cl\, S$ //.

ii). $cl(int\, S) \subseteq cl\, S$ because $int\, S \subseteq S$.

To show $cl\, S \subseteq cl(int\, S)$:

Let $x \in cl\, S$ and let $y \in int\, S$

For $0 < \lambda < 1$, $\lambda x + (1-\lambda) y \in int\, S$, by theorem.

Take limit as $\lambda \to 1$: $\lambda x \in cl(int\, S)$ //.

iii) $S \subseteq cl\, S$ so $int\, S \subseteq int(cl\, S)$.

Show $int(cl\, S) \subseteq int\, S$:

let $x_1 \in int(cl\, S)$

Since $x_1 \in int(cl\, S)$, $\exists\, \varepsilon > 0$ s.t. $N_\varepsilon(x) \subseteq cl\, S$.

Choose $x_2 \in int\, S$, $x_2 \neq x_1$

Let $y = (1+\Delta)x_1 - \Delta x_2$, where $\Delta = \dfrac{\varepsilon}{2\|x_1 - x_2\|}$.

Then $\|x_1 - y\| = \Delta\|x_1 - x_2\| = \frac{\varepsilon}{2}$,

so $y \in N_\varepsilon(x_1)$, so $y \in cl\, S$.

Also, $x_1 = \frac{1}{1+\Delta} y + \frac{\Delta}{1+\Delta} x_2$.

So, by theorem, $x_1 \in int\, S$. //