Convex Sets

Given $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$,

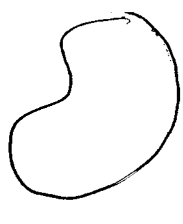
$[x,y]$ denotes the line segment joining x and y

$$[x,y] := \{ z : z = \lambda x + (1-\lambda) y,\ 0 \le \lambda \le 1 \}$$

Def A set C ~~S~~ $\subseteq \mathbb{R}^n$ is called <u>convex</u> if the line segment joining any two points in ~~S~~ the set is also in ~~S~~ C.

Examples: 1)

Convex



Not convex.

2) Subspaces: A point $z \in \mathbb{R}^n$ is said to be a linear combination of ~~a linear combination of x and y~~ the points x and y if $z = \lambda_1 x + \lambda_2 y$, $\lambda_1, \lambda_2 \in \mathbb{R}$.

~~A If $x, y \in \mathbb{R}^n$, a the set of points of the form $\lambda_1 x + \lambda_2 y$~~

~~S is a subspace~~ ~~constitutes the set of linear combinations of x and y~~

A set $S \subseteq \mathbb{R}^n$ is called a <u>subspace</u> of $\mathbb{R}^n$ if ~~any~~ every linear combination of points in S is also in S.

PICTURE

~~2)~~ 3) Affine sets:

A point $z \in \mathbb{R}^n$ is said to be an affine combination of the points x and y if $z = \lambda x + (1-\lambda) y$ for some $\lambda \in \mathbb{R}$.

A set $M \subseteq \mathbb{R}^n$ is called ~~a~~ <u>affine</u> if every affine combination of points in M is also in M. PICTURE

Linear manifolds, flats.

4) ~~Half~~ Half spaces:

$$\{x : a^T x \geq \alpha\}$$

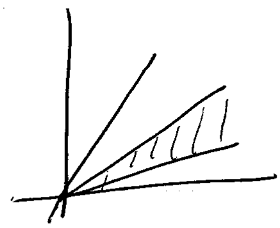
5) Hyperplanes: $\{x : a^T x = \alpha\}$.

6) Cones:
K = a subset of $\mathbb{R}^n$ ~~not~~ closed under positive scalar multiplication

$x \in K, \lambda > 0 \Rightarrow \lambda x \in K$

not necessarily convex

eg



A convex cone is a cone that is a convex set.

Not necessarily pointed:

eg subspaces.

Thm The intersection of an arbitrary collection of convex sets is convex.

ie C_i convex $\forall i \in I \Rightarrow \bigcap\limits_{i \in I} C_i$ is convex

Proof

$$x, y \in \bigcap_{i \in I} C_i \Rightarrow x, y \in C_i \text{ for each } i$$

$$\Rightarrow [x,y] \in C_i \text{ for each } i$$

$$\Rightarrow [x,y] \in \bigcap C_i \qquad //$$

(Index set can be finite or infinite)

Example: Polyhedral sets $\{x : Ax \leq b\}$ are convex.

(Intersection of half spaces).

Define

$$X + Y = \{z \mid \cancel{z}\ z = x + y,\ x \in X,\ y \in Y\}$$

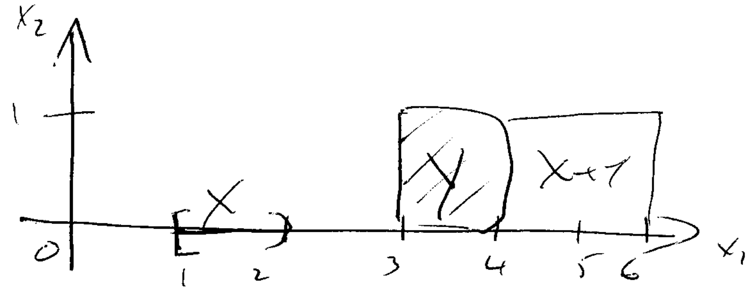
$$X - Y = \{z \mid z = x - y,\ x \in X,\ y \in Y\}$$

Exercise: * If X, Y are convex, then $X+Y$ and $X-Y$ are convex

Example

Note, Need $X+Y$ when talk about separation of two convex sets.)

Example



$$X = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \,\middle|\, 1 \le x_1 \le 2,\ x_2 = 0 \right\}$$

$$Y = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \,\middle|\, 3 \le x_1 \le 4,\ 0 \le x_2 \le 1 \right\}$$

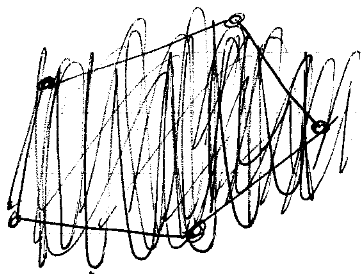
Then $$X+Y = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \,\middle|\, 4 \le x_1 \le 6,\ 0 \le x_2 \le 1 \right\}$$

Defn A point $b \in \mathbb{R}^n$ is called a convex combination of the vectors $a^1, \dots, a^m \in \mathbb{R}^n$ if $\cancel{\exists}$

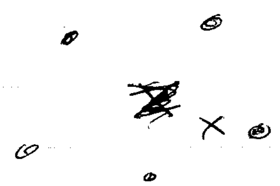
$\exists$ $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$b = \lambda_1 a^1 + \dots + \lambda_m a^m$$

and $\lambda_1 + \dots + \lambda_m = 1$, $\lambda_i \geq 0$



(If take a convex combination, get a point "somewhere in the middle")



THM C is convex $\Leftrightarrow$ every convex combination of points in C is in C.

~~Inf.~~
Algebraically:
$C \subseteq \mathbb{R}^n$ is convex $\Leftrightarrow$

$$(*) \quad \left.\begin{array}{l} x^1, \dots, x^m \in C \\ \lambda_1, \dots, \lambda_m \geq 0 \\ e^T \lambda = 1 \end{array}\right\} \Rightarrow \lambda_1 x^1 + \dots + \lambda_m x^m \in C$$

Pf $\Leftarrow$) Set $m = 2$. NO.

$\Rightarrow$) By induction:

Assume C is convex.

$m = 1$: Clear

$m = 2$: By defn of a convex set

Assume $(*)$ true for m. Show true for $m+1$

Let $x^1, \dots, x^{m+1} \in C$, $\lambda_1, \dots, \lambda_{m+1} \geq 0$, $\Sigma \lambda_i = 1$

We want to show $\lambda_1 x^1 + \dots + \lambda_m x^m + \lambda_{m+1} x^{m+1} \in C$.

Three cases:

i) $\lambda_{m+1} = 0$: Done.

ii) $\lambda_{m+1} = 1$: Then $\lambda_1 = \ldots = \lambda_m = 0$, so $\sum_1^{m+1} \lambda_i x^i = x_{m+1} \in C$

iii) $0 < \lambda_{m+1} < 1$:

Then $\sum_1^m \lambda_i = 1 - \lambda_{m+1} > 0$

Thus

$$\sum_1^{m+1} \lambda_i x^i = \underbrace{\left(\sum_1^m \lambda_i\right) \underbrace{\left[\frac{\lambda_1}{\sum_1^m \lambda_i} x^1 + \cdots + \frac{\lambda_m}{\sum_1^m \lambda_i} x^m\right]}_{\in C \text{ by inductive hypothesis}} + \lambda_{m+1} x^{m+1}}_{\in C \text{ because } (*) \text{ holds for } m=2.}$$

~~By~~

//.