67.611 Mathematical Programming Fall 1988

Lecture 1: 7 Sep 88.

Comments on Outline.

Definition of a nonlinear programming problem:

~~Find an $\bar{x}$, if one exists, such that~~

$$\text{Minimize} \quad f(x)$$

$$\text{subject to} \quad g_i(x) \leq 0, \ i=1,\dots,m \qquad \text{(NLP)}$$

$$h_j(x) = 0, \ j=1,\dots,p$$

$$(x \in X \subseteq \mathbb{R}^n).$$

f is objective function.

Each $g_i(x) \leq 0$ is an inequality constraint

Each $h_j(x)=0$ is an equality constraint.

A vector $x \in X$ is called feasible if it satisfies all the constraints.

We denote the set of feasible solutions by S.

Definitions of optimality:
Let $\bar{x}$ be a feasible point.
$\bar{x}$ is a

<u>global minimizer</u> for (NLP) if $x \in S, x \neq \bar{x}$

$$\Rightarrow f(x) \geq f(\bar{x})$$

~~(strict?)~~
<u>strict global minimizer</u> for (NLP) if $x \in S, x \neq \bar{x}$

$$\Rightarrow f(x) > f(\bar{x})$$

<u>local minimizer</u> for (NLP) if there exists $\varepsilon > 0$

such that $x \in S, x \neq \bar{x}, \|x - \bar{x}\| \leq \varepsilon$

$$\Rightarrow f(x) \geq f(\bar{x})$$

<u>strict local minimizer</u> ——— ———

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Optimality conditions are for <u>local</u> minimizers, and ~~if~~ if we refer to a minimizer, we mean a local minimizer.

Notation:

All vectors in $\mathbb{R}^n$ are column vectors (distinguish $\underset{\text{inner}}{x^T y}$ and $\underset{\text{outer}}{uv^T}$).

Norms in $\mathbb{R}^n$ are Euclidean unless otherwise stated.

For ~~matr~~ $m \times n$ matrix A, usually use operator norm, ie view A as a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

$$= \sqrt{\max \{ x^T A^T A x \} \,|\, \|x\| = 1}$$

$$= \sqrt{\max \text{ e value of } A^T A}$$

$n \times n$ matrix A is symmetric if $a_{ij} = a_{ji}$

positive definite if $x^T A x > 0 \ \forall x \neq 0$

positive semidefinite if $x^T A x \geq 0 \ \forall x$

orthogonal if $A^{-1} = A^T$.

I identity matrix

e vector of ones

e_i ith unit vector.

$$g(x) := \begin{pmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{pmatrix} \qquad h(x) := \begin{pmatrix} h_1(x) \\ \vdots \\ h_p(x) \end{pmatrix}$$

Jacobian:

$$\nabla h = \begin{pmatrix} \nabla h_1(x)^T \\ \vdots \\ \nabla h_m(x)^T \end{pmatrix}$$

Examples

1. Linear Programming

$$\min \quad c^T x$$

$$Ax = b$$
$$x \geq 0$$

Algorithms:
Simplex
Karmarkar

~~Special~~ Special case of NLP.
Will deal with it in a lot more ~~detail~~ detail later.

2. Quadratic Programming

$$\min \quad c^T x + \tfrac{1}{2} x^T H x$$
$$\text{s.t.} \quad A^T x \leq b$$

over /

eg: Portfolio selection.

Have n stocks

jth has expected yield d_j

covariance matrix H

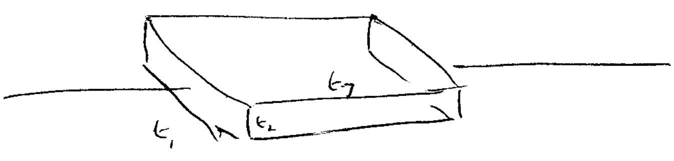
min variance of portfolio
subject to achieving some minimum yield

So $\min \frac{1}{2} x^T H x$

$$\text{s.t.} \quad d^T x \geq \delta$$
$$e^T x = 1$$
$$x \geq 0.$$

3. Engineering design

Transportation of gravel from one side of river to the other



$$\max \quad t_1 t_2 t_3$$

$$\text{s.t.} \quad t_1 t_2 \leq 10$$
$$t_2 \geq 2$$
$$t_1, t_2, t_3 \geq 0.$$

~~Don't need~~
Could make nonnegativity conditions strict.

Let $x_j = \log t_j$. Then:

$$\begin{aligned} \max \quad & \exp(x_1 + x_2 + x_3) \\ \text{s.t.} \quad & \exp(x_1 + x_2) \leqslant 10 \\ & \exp(x_2) \geqslant 2 \end{aligned}$$

Since log is a monotone function, can take logs:

$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 \leqslant \log 10 \\ & x_2 \geqslant \log 2 \end{aligned}$$

So have linear program.