DS95

SEQUENCES & SUMMATIONS (§2.4)

DEFN A SEQUENCE is a function from a subset of the set of integers (usually either $\{0,1,2,\ldots\}$ or $\{1,2,3,\ldots\}$) to a set S.

The TERM a_n is the image of the integer n.

Eg: $a_1 = 3$ $a_2 = 6$ $a_3 = 12$ $a_4 = 24$ ...

so $a_n = 3 \cdot 2^{n-1}$ GEOMETRIC PROGRESSION

Eg: Special integer sequences: see Nick Sloane's On-line encyclopedia of integer sequences (Handout.)

Summations:

Thm If a and r are real numbers and $r \neq 0$, then $\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r-1}$

Proof Let $S = \sum_{j=0}^{n} ar^j$.

Then $rS = \sum_{j=0}^{n} ar^{j+1} = ar^{n+1} + \sum_{j=0}^{n} ar^j - a$

$= ar^{n+1} + S - a$

Rearranging: $(r-1)S = ar^{n+1} - a$

so $S = \frac{ar^{n+1} - a}{r-1}$ $\boxed{//}$

CARDINALITY

DEFN The sets A and B have the same CARDINALITY if and only if there is a one-to-one correspondence from A to B.

DEFN A set that is finite or has the same cardinality as the set of positive integers is called COUNTABLE. If an infinite set is countable, write $|S| = \aleph_0$. A set that is not countable is UNCOUNTABLE.

aleph null

first letter of Hebrew alphabet.

Eg: The set of squares is countable:

$$1 \to 1, \quad 2 \to 4, \quad 3 \to 9, \quad 4 \to 16, \ldots$$

Eg: The set of positive rational numbers is countable.

For proof see text.

Eg: The set of positive real numbers is uncountable.

Proof: see text.

Recursive Definitions & Structural Induction (§4.3)

Recursively defined functions:

BASIS STEP: Specify the value of the function at zero.

RECURSIVE STEP: Give a rule for finding its value at an integer from its values at smaller integers.

Eg: Fibonacci Numbers:

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}$$

So $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, $f_6 = 8$, $f_7 = 13$, $f_8 = 21, \ldots$

Let $\alpha = \frac{1+\sqrt{5}}{2}$, the GOLDEN RATIO. $\alpha \approx 1.618$

α solves $\alpha^2 - \alpha - 1 = 0$. Also, $\frac{1}{\alpha} = \alpha - 1$.

THM If $n \geq 2$ is even then $\frac{f_n}{f_{n-1}} < \alpha$. If $n \geq 3$ is odd then $\frac{f_n}{f_{n-1}} > \alpha$

Proof Base case: $\frac{f_2}{f_1} = \frac{1}{1} = 1 < \alpha \qquad \frac{f_3}{f_2} = \frac{2}{1} = 2 > \alpha$

Inductive Step:

$$\frac{f_n}{f_{n-1}} = \frac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \frac{f_{n-2}}{f_{n-1}} = 1 + 1 \Big/ \left(\frac{f_{n-1}}{f_{n-2}}\right)$$

If n even: $\frac{f_n}{f_{n-1}} = 1 + \frac{1}{\left(\frac{f_{n-1}}{f_{n-2}}\right)} < 1 + \frac{1}{\alpha} = 1 + (\alpha - 1) = \alpha$

If n odd: $\frac{f_n}{f_{n-1}} = 1 + \frac{1}{\left(\frac{f_{n-1}}{f_{n-2}}\right)} > 1 + \frac{1}{\alpha} = 1 + (\alpha - 1) = \alpha$ ▱

Recursively defined sets & structures

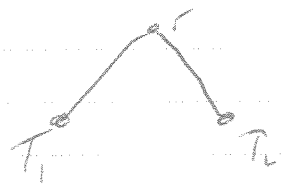
Eg: The set of full BINARY TREES can be defined recursively:

Basis Step: Full binary tree consisting of a single vertex, the root r.

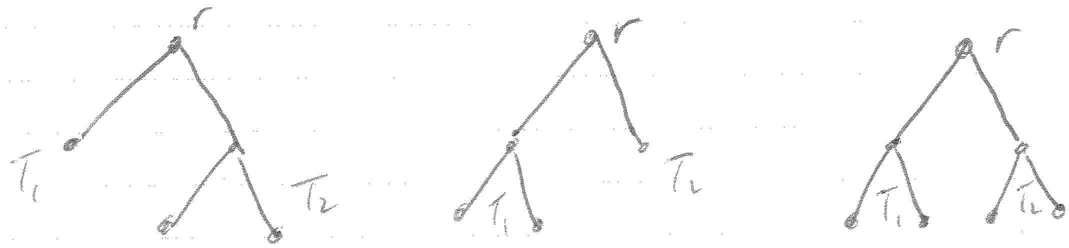
Recursive Step: If T_1 and T_2 are disjoint full binary trees, there is a full binary tree, denoted by $T_1 \cdot T_2$, consisting of a root r together with edges connecting the root ~~of each~~ to each of the roots of the left subtree T_1 and the right subtree T_2.

Basis step:

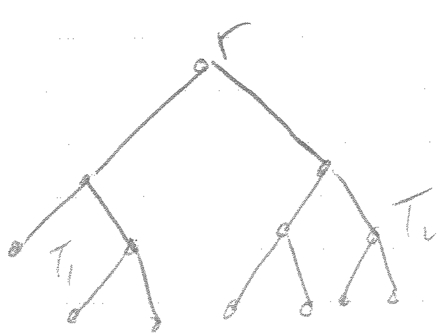
Step 1:



Step 2:



Step 3:



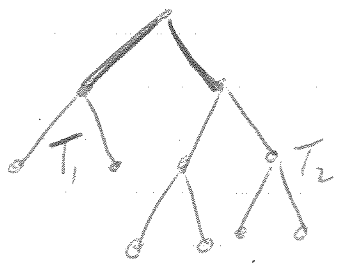
etc

DEFN The HEIGHT $h(T)$ of a full binary tree T:

BASIS STEP: If T only contains a root node then $h(T) = 0$

RECURSIVE STEP: If $T = T_1 \cdot T_2$ then $h(T) = 1 + \max(h(T_1), h(T_2))$

E.g.



$h(T_1) = 1$
$h(T_2) = 2$
$h(T) = 3$

after theorem, come back here:

	$2^{h(T)+1}-1$
$n(T_1) = 3$	3
$n(T_2) = 7$	7
$n(T) = 11$	15

Let $n(T)$ = number of vertices of full binary tree.

Theorem $n(T) \leq 2^{h(T)+1} - 1$.

$2^{h(T)+1} - 1 = 1$

Proof See text for details. Use ~~str~~ structural induction.

Basis step: If only have root node then $h(T) = 0$, $n(T) = 1$

~~Show~~ Have $n(T) = 1 + n(T_1) + n(T_2)$

$\leq 1 + 2\max(n(T_1), n(T_2))$

h increases by 1, while $n(T)$ goes up by a factor of 2.