Chapter 6 Discrete Probability

Introduction (§6.1)

Defn If S is a finite sample space of equally likely outcomes, and E is an event (that is, a subset of S) then the <u>probability</u> of E is $p(E) = \frac{|E|}{|S|}$.

Eg: Roll two dice. What is probability they total 5?

Answer: $|S| = 6 \times 6 = 36$

$|E|$:

dice 1	dice 2
1	4
2	3
3	2
4	1

4 answers, so $|E| = 4$

$\therefore p(E) = \frac{4}{36} = \frac{1}{9}$.

Eg: Blackjack: What is the probability of being dealt 21?

Get two cards. Ace worth 11, KQJ10 all worth 10.

So need Ace + one of KQJ10

$|E| = \binom{4}{1}\binom{16}{1}\binom{2}{1} = 128$ $\qquad |S| = \binom{52}{2} = \frac{52 \cdot 51}{2} = 26 \cdot 51$

So $p(E) = \frac{128}{26 \cdot 51} \approx 0.0965$

Eg: Urn containing balls labelled $1, 2, \ldots, 10$.

What is prob that the balls $3, 9, 7$ are drawn in that order, if the selected ball is not returned to the urn?

Answer: # 3 ball drawings is $10 \cdot 9 \cdot 8$ ~~since~~ (note the ordering is important, so don't divide by $3!$)

$= 720$

SAMPLING WITHOUT REPLACEMENT

Our ordering occurs in only one way.

So $\text{Prob}(E) = \frac{1}{720}$.

Now: ~~Dra~~ What is prob that $3, 9, 7$ are drawn in any order, if the selected ball is not returned?

Now $|E| = 3! = 6$ so $\text{Prob}(E) = \frac{6}{720} = \frac{1}{120}$.

SAMPLING WITH REPLACEMENT.

Now: In our order, if balls are replaced after drawing?

$|E| = 1$ $\quad |S| = 10^3$, so $\text{Prob}(E) = \frac{1}{1000}$.

If our balls can be drawn in any order, with the drawn ball being replaced:

$|E| = 6$, $\quad |S| = 10^3$, so $\text{Prob}(E) = \frac{6}{1000} = 0.006$.

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Combination of Events:

THM Let E be an event in a sample space S. The probability of the event $\bar{E}$, the complementary event of E, is given by $p(\bar{E}) = 1 - p(E)$.

Proof. Note that $|\bar{E}| = |S| - |E|$.

$$\text{So} \quad p(\bar{E}) = \frac{|S| - |E|}{|S|} = 1 - p(E). \quad \square$$

Eg: 5 people from Upstate, 6 from Downstate. Form a committee of four people randomly. What is the committee contains at least one person from Downstate?

Use complement:

P(at least one from downstate) = 1 − P(none from downstate)

$$= 1 - \frac{\binom{5}{4}}{\binom{11}{4}}$$

— choose all four members from upstate

— ways to choose 5 members

$$= 1 - \frac{5}{\left(\frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2 \cdot 1}\right)}$$

$$= 1 - \frac{1}{666}$$

$$= \frac{665}{666}.$$

Eg: Birthday Problem

Assume all 366 days are equally likely as a birthday.

~~What is pr~~

How many people are needed in a group to ensure $p(\text{two share a birthday}) \geq \frac{1}{2}$.

Answer:

$P(\text{two people share a birthday})$

$$= 1 - P(\text{all birthdays different})$$

$$= 1 - \underbrace{\frac{365}{366} \cdot \frac{364}{366} \cdots \cancel{364} \cdots \frac{367-n}{366}}_{\substack{\text{have } n \text{ people in group,} \\ n-1 \text{ terms.}}}$$

$$\approx \begin{cases} 0.475 & \text{if } n = 22 \\ 0.506 & \text{if } n = 23. \end{cases}$$

Thm Let E_1 and E_2 be two events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Proof $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$

So $p(E_1 \cup E_2) = \frac{|E_1|}{|S|} + \frac{|E_2|}{S} - \frac{|E_1 \cap E_2|}{S} = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

Eg: Roll two dice. What is probability that ~~they add up to 5 or~~ one of the dice is a 3?

~~Answer E_1: add [illegible] total 5. $p(E_1) = \frac{1}{9}$. $(= \frac{4}{36})$~~

~~E_2: roll a~~ 3.

Answer: E_1: first dice is a 3. $p(E_1) = \frac{1}{6}$

E_2: second dice is a 3. $p(E_2) = \frac{1}{6}$

$E_1 \cap E_2$: both dice are 3s. $p(E_1 \cap E_2) = \frac{1}{36}$

So $p(E_1 \cup E_2) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{12}{36} - \frac{1}{36} = \underline{\underline{\frac{11}{36}}}$

Eg: Roll two dice. What is probability that they add up to 5, or one of the dice is a 3?

E_1: add up to 5. $p(E_1) = \frac{4}{36}$ E_2: one of the dice is a 3. $p(E_2) = \frac{11}{36}$

$E_1 \cap E_2$: add up to 5, and one of the dice is a 3: $\frac{2}{36}$ (either 2+3 or 3+2)

So $p(E_1 \cup E_2) = \frac{4}{36} + \frac{11}{36} - \frac{2}{36} = \frac{13}{36}$.

Blackjack again

Player has 16, Dealer is showing 10 as face up card.
Dealer hits as long as has ≤ 16.

Hit or Stand?

Assume using multiple decks, so each card is still equally likely.

$$p(\text{dealer} \geq 17 \text{ already}) = \frac{8}{13} \qquad (7, 8, 9, 10, J, Q, K, A)$$

$$p(\text{player busts immediately}) = \frac{8}{13}. \qquad (6, 7, 8, \ldots, Q, K)$$

So hitting is at least as good as standing, even if only look one card ahead.
Need to consider dealer taking more than one card, so complicated.

If STAND:

$$P(\text{lose}) \geq P(\text{dealer} \geq 17 \text{ already}) = \frac{8}{13}$$

If HIT

$$P(\text{lose}) = P(\text{bust immediately}) + \frac{1}{13}P(\text{lose with } 17 \text{ when dealer has } 10)$$

$$+ \frac{1}{13}P(\text{lose with } 18 \mid \text{dealer has } 10)$$
$$+ \ldots + \frac{1}{13}P(\text{lose with } 20 \mid \text{dealer has } 10)$$

PROBABILITY THEORY (§6.2)

S: sample space with a finite or countable number of outcomes.

$p(s)$: probability of outcome s.

Need. (i) $0 \le p(s) \le 1 \quad \forall s \in S$

(ii) $\sum_{s \in S} p(s) = 1$.

p: PROBABILITY DISTRIBUTION.

Eg: Biased ~~coin~~ coin. Three times as likely to have heads as tails.

$p(H)$? $p(T)$?

$$\left.\begin{array}{l} p(H) + p(T) = 1 \\ p(H) = 3\,p(T) \end{array}\right\} \Rightarrow 4p(T) = 1 \Rightarrow p(T) = \tfrac{1}{4},\ p(H) = \tfrac{3}{4}.$$

Eg: Prob of at least one head in two rolls with this dice:

$$p(H \text{ then } T) + p(T \text{ then } H) + p(H \text{ then } H)$$
$$= \tfrac{3}{4}\cdot\tfrac{1}{4} + \tfrac{1}{4}\cdot\tfrac{3}{4} + \tfrac{3}{4}\cdot\tfrac{3}{4} = \tfrac{15}{16}$$

Also $= 1 - \underbrace{p(T \text{ then } T)}_{\text{Complementary event}} = 1 - \tfrac{1}{4}\cdot\tfrac{1}{4} = \tfrac{15}{16}$

Conditional probability

Eg: P(biased coin has at least two heads in three flips)

given that the first flip is a head

$= P$(at least one head in last two flips)

$= \frac{15}{16}.$

Defn Let E and F be events with $p(F) > 0$. The CONDITIONAL PROBABILITY of E given F (denoted $p(E|F)$) is defined as:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

Returning to our example

$$p(F) = \frac{3}{4}.$$

$$p(E \cap F) = p(HHH) + p(HHT) + p(HTH)$$

$$= \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) = \frac{27+9+9}{64} = \frac{45}{64}.$$

So $\frac{p(E \cap F)}{p(F)} = \frac{45/64}{3/4} = 15/16$ ✓.

INDEPENDENCE

DEFN The events E and F are INDEPENDENT if and only if $P(E \cap F) = p(E)p(F)$

Eg: $p(H \text{ then } T) = p(H)p(T) = \frac{3}{16}$.

Eg: Fair coin. Toss three times. E: get even number of heads
F: first toss is a tail.

$$p(E) = p(0 \text{ heads}) + p(2 \text{ heads}) = \frac{1}{8} + \underset{HHT,\ HTH,\ THH}{\frac{3}{8}} = \frac{1}{2}$$

$p(F) = \frac{1}{2}$. So $p(E)p(F) = \frac{1}{4}$

$p(E \cap F) = \cancel{p(HHT) + p}\ p(TTT) + p(THH) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

So events are independent.

Because: $p(\text{even number}) = \frac{1}{2}$, for any number of tosses.

Goes back to Corollary 2 in §5.4 (binomial coefficients):

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

The negative terms correspond to odd subsets
The positive terms correspond to even subsets.

(Example of binomial distribution, see next page)

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Bernoulli Trials & Binomial Distribution

Have two possible outcomes success or failure. $P(\text{failure}) = 1 - P(\text{success})$

Eg flip a coin. $P(T) = 1 - p(H)$

Say $p(H) = \frac{3}{4}$.

$p($ ~~six~~ three heads in ~~8~~5 flips$)$

$$= \binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$$

$$= \frac{270}{4^5} = \frac{270}{512} = \frac{135}{256}$$

Let p = probability of success

Probability of k successes in n trials is denoted $b(k; n, p)$,

and $b(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$

BINOMIAL DISTRIBUTION

Random Variables

<u>Defn</u> A RANDOM VARIABLE is a <u>function</u> from the sample space of an experiment to the set of real numbers. A random variable assigns a real number to each possible outcome.

<u>Eg:</u> Toss a coin five times.
Random variable = # heads.

$\left.\begin{array}{l} t = \text{instance}. \quad X(t) = \text{random variable}. \\ X(TTHTH) = 2, \; X(HHHHT) = 4, \ldots \end{array}\right.$

<u>Eg:</u> Give an exam to 100 students.
Random variable = # students getting at least 90%.

<u>Defn</u> The DISTRIBUTION of a random variable X on a sample space S is the set of pairs $(r, p(X=r))$ for all $r \in X(S)$, where $p(X=r)$ is the probability that $X=r$

<u>Eg:</u> Toss a biased coin 3 times. $p(H) = \frac{3}{4}$, $p(T) = \frac{1}{4}$.

$X(t)$ = # heads

$$\left.\begin{aligned} p(X=0) &= \left(\tfrac{1}{4}\right)^3 = \tfrac{1}{64} \\ p(X=1) &= \binom{3}{1}\left(\tfrac{3}{4}\right)^1\left(\tfrac{1}{4}\right)^2 = \tfrac{9}{64} \\ p(X=2) &= \binom{3}{2}\left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) = \tfrac{27}{64} \\ p(X=3) &= \binom{3}{3}\left(\tfrac{3}{4}\right)^3\left(\tfrac{1}{4}\right)^0 = \tfrac{27}{64} \end{aligned}\right\} \quad \text{Check: } \tfrac{1}{64} + \tfrac{9}{64} + \tfrac{27}{64} + \tfrac{27}{64} = 1 \checkmark.$$

Bayes Theorem (§6.3)

Eg: Two plates of cookies.

Plate A: 5 chocolate chip, 15 plain

Plate B: 10 chocolate chip, 5 plain.

Miguel picked a plate at random and then picked a cookie from that plate at random.

Miguel picked a chocolate chip cookie. What is the probability he picked from Plate A?

Event A: pick from plate A

Event C: pick a chocolate chip cookie.

Want to find $\cancel{p(A)}$ $p(A|C)$.

$$p(A|C) = \frac{p(A \cap C)}{p(C)}$$

Now, $p(C|A) = \dfrac{p(A\cap C)}{p(A)}$, so $p(A \cap C) = p(C|A)\,p(A)$.

Also, $p(C) = p(C|A)\,p(A) + p(C|\bar{A})\,p(\bar{A})$

$\left(= p(A \cap C) + p(\bar{A} \cap C)\right)$

So: $$p(A|C) = \frac{p(C|A)\,p(A)}{p(C|A)\,p(A) + p(C|\bar{A})\,p(\bar{A})}$$

$$= \frac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{2}} \cancel{= \frac{3}{3+8} = \frac{3}{11}} = \frac{3}{3+8} = \frac{3}{11}.$$

Bayes Theorem E, F events from sample space S with $p(E) \neq 0$ and $p(F) \neq 0$.

Then $$p(F|E) = \frac{p(E|F)\,p(F)}{p(E|F)p(F) + p(E|\bar{F})p(\bar{F})} = \frac{p(E|F)\,p(F)}{p(E)}.$$

Eg: Medical testing:

Disease affects 1 person in 1000.
Have blood test for the disease.
If have disease, test shows presence of disease 100% of time
If don't have disease, test shows presence of disease 1% of time (FALSE POSITIVE)

If a person has the test, and it comes back positive, what is the probability they have the disease?

Event T : test is positive
Event D : have the disease.
Want to find $p(D|T)$.

$$p(D|T) = \frac{p(T|D)\,p(D)}{p(T|D)p(D) + p(T|\bar{D})p(\bar{D})}$$

$$= \frac{1 \cdot \frac{1}{1000}}{1 \cdot \frac{1}{1000} + \frac{1}{100} \cdot \frac{999}{1000}} = \frac{100}{100 + 999} \approx \frac{1}{11}.$$

So less than 10% chance actually have the disease.

Eg: Spam Filter

An email contains the word OEM. What is the probability that it is spam?

Need to know: occurrence of OEM in spam & non-spam messages.

Assume: Have 5000 spam messages. "OEM" occurs in 500 of them
Have 3000 non-spam messages. "OEM" occurs in 10 of them.

Events: S: email is spam
O: email contains OEM

$$p(S|O) = \frac{p(O|S)\,p(S)}{p(O|S)\,p(S) + p(O|\bar{S})\,p(\bar{S})}$$

(a) $$p(S|O) = \frac{\frac{500}{5000}\cdot\frac{5000}{8000}}{\frac{500}{5000}\cdot\frac{5000}{8000} + \frac{10}{3000}\cdot\frac{3000}{8000}}$$ (with prior: $p(S) = \frac{5000}{8000}$)

$$= \frac{500}{510} \approx 0.98$$

(b) $$p(S|O) = \frac{\frac{500}{5000}\cdot\frac{1}{2}}{\frac{500}{5000}\cdot\frac{1}{2} + \frac{10}{3000}\cdot\frac{1}{2}}$$ (without prior, ie with a uniform prior)

$$\cancel{= \frac{1500}{1500+150}}$$

$$= \frac{300}{300+10} = \frac{300}{310} \approx 0.968$$

Eg: <u>Spam filter with two words:</u>

Words "guarantee" and "offer" both appear. E_1: guarantee
E_2: offer

Assume these words appear independently in spam & non-spam,
i.e., $(E_1|S)$ and $(E_2|S)$ are independent events,
and E_1 and E_2 are independent.

Then $$p(S|E_1 \cap E_2) = \frac{p(E_1 \cap E_2|S)\,p(S)}{p(E_1 \cap E_2|S)\,p(S) + p(E_1 \cap E_2|\bar{S})}$$

$$= \frac{p(E_1|S)\,p(E_2|S)\,p(S)}{p(E_1|S)\,p(E_2|S)\,p(S) + p(E_1|\bar{S})\,p(E_2|\bar{S})\,p(\bar{S})}$$

from independence assumption.

Eg, if $p(E_1|S) = \frac{1}{100}$, $p(E_2|S) = \frac{3}{100}$, $p(S) = p(\bar{S}) = \frac{1}{2}$,
$p(E_1|\bar{S}) = \frac{1}{1000}$, $p(E_2|\bar{S}) = \frac{2}{1000}$

then $$p(S|E_1 \cap E_2) = \frac{\frac{1}{100}\cdot\frac{3}{100}\cdot\frac{1}{2}}{\frac{1}{100}\cdot\frac{3}{100}\cdot\frac{1}{2} + \frac{1}{1000}\cdot\frac{2}{1000}\cdot\frac{1}{2}}$$

$$= \frac{300}{302} \approx 0.993.$$

EXPECTED VALUE & VARIANCE (§6.4) (NOT ON TEST)

DEFN Expected value of random variable $X(s)$ on sample space S is

$$E(X) = \sum_{s \in S} p(s) X(s)$$

Eg: Draw card in blackjack. Say A is worth 1.
What is expected value of drawn card?

$$E(X) = \frac{1}{13} \cdot 1 + \frac{1}{13} \cdot 2 + \dots + \frac{1}{13} \cdot 9 + \frac{4}{13} \cdot 10$$

$$= \frac{1}{13}\left((1+2+\dots+9) + 40\right)$$

$$= \frac{1}{13}(45+40) = \frac{85}{13} \approx 6.54.$$

DEFN X random variable on sample space S. VARIANCE of X is

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s)$$

STANDARD DEVIATION is $\sigma(X) = \sqrt{V(X)}$

THM $V(X) = E(X^2) - (E(X))^2$

Eg: Blackjack draw:

$$E(X^2) = \sum_{k=1}^{9} \frac{k^2}{13} + \frac{4}{13} \cdot 100 = \frac{1}{13}\left(\overbrace{\tfrac{1}{6}(9)(10)(19)}^{\sum_1^n k^2 = \frac{1}{6}n(n+1)(2n+1)} + 400\right)$$

$$= \frac{1}{13}(685) \approx 52.7 \qquad \sigma(X) \approx 7.26$$