DS48

Mathematical Induction (§4.1)

To prove $P(n)$ is true for all positive integers n:

1. BASIS STEP: Verify that $P(1)$ is true
2. INDUCTIVE STEP: Show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

Eg Show $1+2+\dots+n = \frac{n(n+1)}{2}$ $\forall$ positive integers n.

Basis step: $n=1$: $1+2+\dots+n = 1$.
$\frac{n(n+1)}{2} = \frac{1(2)}{2} = 1$ ✓

Inductive step: Assume true for $n=1,\dots,k$.
Show true for $n=k+1$:

$1+2+\dots+k+(k+1) = \frac{k(k+1)}{2} + k+1$ by inductive hypothesis

$= \frac{1}{2}(k^2 + k + 2k + 2)$

$= \frac{1}{2}(k^2+3k+2)$

$= \frac{1}{2}(k+1)(k+2)$

$= \frac{(k+1)((k+1)+1)}{2}$

To prove $P(n)$ is true for all positive integers ~~≥~~ $n \geqslant b$:

Need to change Basis Step: Verify that $P(b)$ is true.

Inductive Step as before.

eg: From first class, we showed $2^n < n!$ for $n \geqslant 4$.

Basis Step: If $n=4$ then $2^n = 16$, $n! = 24$

Inductive Step: Assume true for $n=k$, show true for $n=k+1$:

$$\frac{2^{k+1}}{(k+1)!} = \frac{2^k}{k!} \cdot \frac{2}{(k+1)}$$

$$< \frac{2^k}{k!} \quad \text{since } k \geqslant 4$$

$$< 1 \quad \text{from inductive hypothesis.}$$

Eg. Show: $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{(n-1)n} = \frac{n-1}{n}$ for $n \geq 2$.

Base case: $n=2$: $\frac{1}{1.2} = \frac{1}{2}$ ✓

Inductive step: Assume true for $n \leq k$.

$n=k+1$:

$$\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{(k-1)k} + \frac{1}{k(k+1)} = \frac{k-1}{k} + \frac{1}{k(k+1)} = \cancel{\frac{k+}{k(k+1)}}$$

$$= \frac{k^2-1+1}{k(k+1)} = \frac{k}{k+1} \quad ✓.$$

Eg. Show $\sum_{i=0}^{n} r^{i} = \frac{r^{n+1}-1}{r-1}$ provided $r \neq 1$.

Base case: $n=0$: $\sum_{i=0}^{n} r^{i} = r^0 = 1 = \frac{r^{0+1}-1}{r-1}$ ✓

Inductive step: Assume true for $n \leq k$.

$n=k+1$:

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1}$$

$$= \frac{r^{k+1}-1}{r-1} + r^{k+1}$$

$$= \frac{r^{k+1}-1+r^{k+2}-r^{k+1}}{r-1}$$

$$= \frac{r^{k+2}-1}{r-1} \quad ✓$$

Results about sets

Eg: A set with n elements has 2^n subsets.

Proof Base case : $n = 0$. Only subset is ϕ, and $2^0 = 1$ ✓
Also: $n = 1$: Subsets are ϕ and the element itself, and $2^1 = 2$ ✓

Inductive step:
Add one element to a set with k elements.

Label the elements $\{1, 2, \ldots, k, k+1\}$.

All 2^k subsets of $\{1, 2, \ldots, k\}$ are subsets of $\{1, 2, \ldots, k+1\}$

~~$\{k+1\}$ is a subset.~~

Any subset of $\{1, 2, \ldots, k\}$ (including ϕ) + element $k+1$ is a subset of $\{1, \ldots, k+1\}$

So # subsets of $\{1, 2, \ldots, k+1\}$ is $2(2^k) = 2^{k+1}$ ✓.

Eg: Extension of One of De Morgan's Laws:

$$\bigcap_{j=1}^{n} \overline{A_j} = \overline{\bigcup_{j=1}^{n} A_j}$$ when $A_1, A_2, \ldots, A_n \subseteq U$, a universal set, and $n \geq 2$

Proof:

Base case: $n=2$:

This is $\overline{A_1} \cap \overline{A_2} = \overline{A_1 \cup A_2}$, one of De Morgan's Laws.

Inductive step:

Assume true for $n \leq k$.

For $n = k+1$:

$$\bigcap_{j=1}^{k+1} \overline{A_j} = \left(\bigcap_{j=1}^{k} \overline{A_j}\right) \cap \overline{A_{k+1}}$$

$$= \left(\overline{\bigcup_{j=1}^{k} A_j}\right) \cap \overline{A_{k+1}}$$ by inductive hypothesis

$$= \overline{\left(\bigcup_{j=1}^{k} A_j\right) \cup A_{k+1}}$$ by De Morgan's Law

$$= \overline{\bigcup_{j=1}^{k+1} A_j}$$

$\blacksquare$

STRONG INDUCTION ~~& Well Ordering~~ (§4.2)

Eg: Show that if n is an integer greater than 1 then n can be written as the product of primes.

Proof: Basis Step: $n=2$: ✓

Inductive Step:

STRONG INDUCTION: Assume true for $n = 2, 3, \ldots, k-1, k$.
Show true for $n = k+1$.

If $k+1$ is a prime then we are done.

Else, k is a product of two smaller numbers, $k+1 = pq$.

By the strong inductive hypothesis, p and q are each expressible as the product of primes.

Thus, $k+1$ can be written as a product of primes. ∎

Eg: Have n stones.
Two players take it in turns to remove 1 or 2 stones.
The player removing the last stone wins.

Then: if there are $3j$ stones then player 2 wins,
else player 1 wins, where j is a ~~____~~ positive integer.

<u>Proof</u>: Basic step:
If $n=1$: Player 1 removes the stone and wins
If $n=2$: Player 1 removes both stones and wins
If $n=3$: Player 1 removes one or two stones. Player 2 removes the remaining stones and wins.

Strong Inductive Step:
Assume have $k+1$ stones, and know outcome of have $1, 2, 3, \ldots, k-1, k$ stones.

If $k+1 \neq 3j$:
Player one removes enough stones to leave exactly $3j$ for some integer j.
By the inductive hypothesis, the next player to remove a stone loses.
Ie, player 2 loses, and player 1 wins.

If $k+1 = 3j$:
If player 1 removes p stones then player 2 removes $3-p$ stones,
and we have $3(j-1)$ stones left, with player 1 to play.
By inductive hypothesis, player 1 loses.