MATH 2800 INTRO TO DISCRETE STRUCTURES — SPR 2007.

Text: Rosen, Discrete Math, 6th Edition.

Cover chapters 1–9.

Quick Overview of Topics.

Induction

Eg: Show $2^n < n!$ for $n \geq 4$.

Proof: $2^4 = 16$, $4! = 24$, so true for $n = 4$.

Assume true for some integer k, so $2^k < k!$

Look at $n = k+1$ $\quad 2^{k+1} = 2(2^k)$

$(k+1)! = k(k!)$

Since $k > 2$ and $2^k < k!$, we have $2^{k+1} < (k+1)!$

Counting

Eg: How many five card poker hands contain a full house (three of one kind, two of another)?

Answer: $52 \cdot 72$ $\left(= 13 \binom{4}{3} \cdot 12 \binom{4}{2} \right)$.

DS 2.

Probability

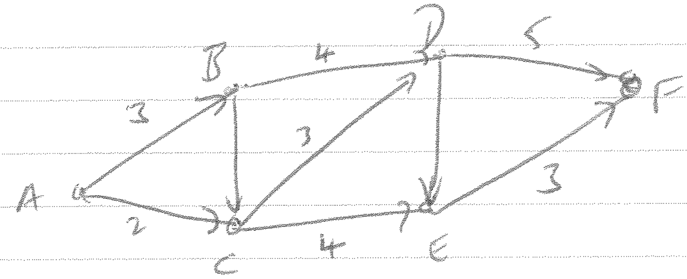
Eg: Given an urn with 3 red balls and 5 white balls, if we pick two balls at random, what is the probability that we get two red balls? $\frac{3}{28}$ $\left(= \frac{\binom{3}{2}}{\binom{8}{2}} = \frac{3}{8} \right.$

Eg: What is the probability that two people in this room share a birthday?

Eg: What is the probability that a certain email is spam?

Graphs

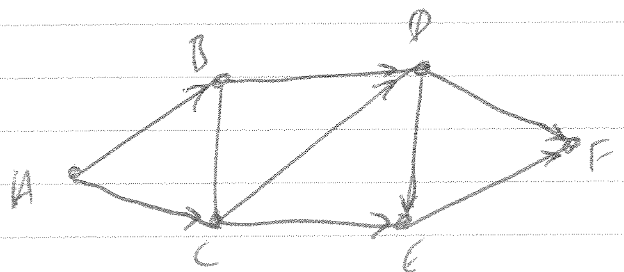
Eg:



What is the shortest path from A to F?

(Eg: packet routing.)

Eg:



Is there a path from A to F that visits each node exactly once?

(Eg: UPS delivery.)

SETS (Section 2.1)

DEFN (a) A SET is an unordered collection of objects
(b) The objects are called ELEMENTS or MEMBERS of the set.

Eg: ① The people in this room

② NFL teams

③ The courses you are taking.

④ ~~___~~ Square numbers less than 30: = $\{1, 4, 9, 16, 25\}$. (Notation!)

DEFN The set A is a SUBSET of B if and only if every element of A is also a member of B. Write $A \subseteq B$.

Eg: The set of NFL playoff teams this season is a subset of all NFL teams.

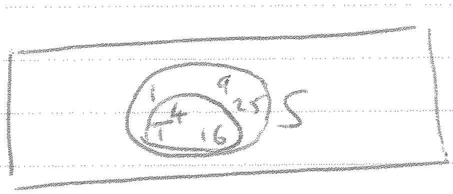
Eg: The EMPTY SET ϕ is a subset of any set.

Eg: Given a set S, the set S is itself a subset of S.

DEFN The ~~cardin~~ CARDINALITY of a set S is the number of elements in S, written $|S|$.

Eg: If $S = \{1, 4, 9, 16, 25\}$ then $|S| = 5$.

Venn diagram:
$S = \{1, 4, 9, 16, 25\}$,
$T = \{4, 16\}$:



UNIVERSAL SET

DEFN The POWER SET of S is the set of all subsets of S. Written $P(S)$

Eg: $S = \{1, 4, 9, 16\}$

$$P(S) = \{\phi, \{1\}, \{4\}, \{9\}, \{16\}, \{1,4\}, \{1,9\}, \{1,16\}, \ldots, \{4,9,16\}, \{1,4,9,16\}\}.$$

Always get $|P(S)| = 2^{|S|}$.

DEFN (a) The ORDERED n-TUPLE $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.

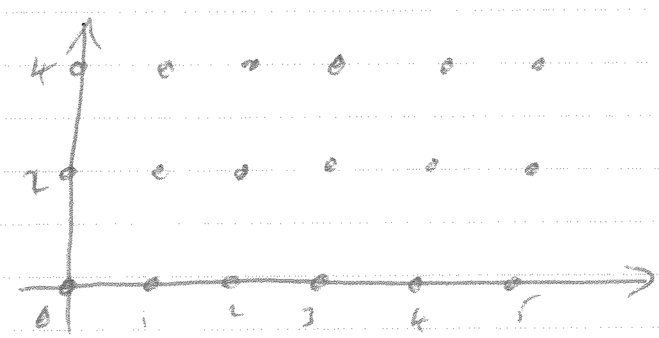
(b) Ordered 2-tuples are called ORDERED PAIRS.

(c) Let A and B be sets. The CARTESIAN PRODUCT of A and B is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
Hence $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$.

Notation

Eg: $A = \{0, 1, 2, 3, 4, 5\}$, $B = \{0, 2, 4\}$

$A \times B$:



$|A \times B| = 18$

$= |A| \times |B|$.

$$A \times B = \{(0,0), (1,0), (2,0), (3,0), (4,0), (5,0), \ldots, (5,4)\}.$$

NOTE Cartesian products of more than two sets defined similarly. Eg: $A \times B \times C$.

SET OPERATIONS (§1.2)

DEFN ① The UNION of two sets A and B is
$$A \cup B = \{x : x \in A \vee x \in B\}$$
$\vee$ means "or".

② The INTERSECTION of two sets A and B is
$$A \cap B = \{x : x \in A \wedge x \in B\}$$
$\wedge$ means "and".

Eg: $A = \{1, 3, 5, 7, 9\}$ $B = \{1, 4, 9\}$

$A \cup B = \{1, 3, 4, 5, 7, 9\}$ $A \cap B = \{1, 9\}$

DEFN ① Two sets A and B are ~~disjoint~~ DISJOINT if $A \cap B = \phi$

② The DIFFERENCE of A and B is $A - B = \{x : x \in A \wedge x \notin B\}$

Eg: ① $A = \{1, 3, 5\}$ $B = \{2, 4, 6\}$ A, B are disjoint

② $A = \{1, 3, 5, 7, 9\}$ $B = \{1, 4, 9\}$

$A - B = \{3, 5, 7\}$

$B - A = \{4\}$

Defn ① The UNIVERSAL SET U contains all objects under consideration

② The COMPLEMENT of a set A is $\bar{A} := U - A$.

Eg: U = positive integers from 1 to 10

$B = \{1, 3, 5, 7\}$. Then $\bar{B} = \{2, 4, 6, 8, 9, 10\}$

Set Identities

The book lists 19 identities you need to know.

We will look at some of them.

① One of De Morgan's Laws: $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof:

$$\begin{aligned}\overline{A \cup B} &= \{x : x \notin A \cup B\} && \text{by definition of complement}\\ &= \{x : x \notin A \wedge x \notin B\} && \text{by definition of union}\\ &= \{x : x \in \bar{A} \wedge x \in \bar{B}\} && \text{by definition of complement}\\ &= \bar{A} \cap \bar{B}. && \text{by definition of intersection}\end{aligned}$$

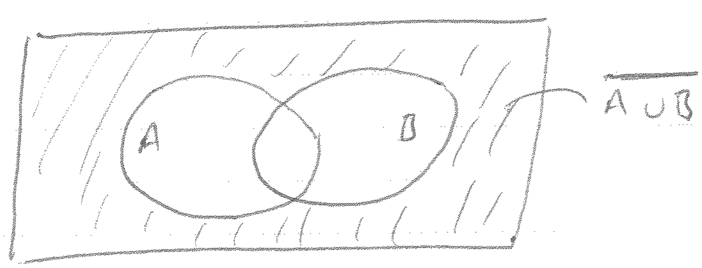
$\blacksquare$

~~See more expansive proof on next page~~

No: do this proof!!

~~Thm~~

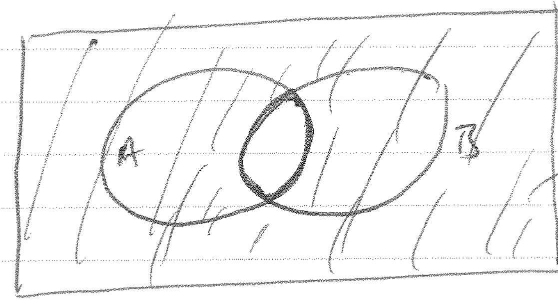
Venn diagram:



The other De Morgan law: $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

For proof, see text.

Venn diagram:



$\overline{A \cap B}$.

$\overline{(\bar{A})} = A$.

Proof $\overline{(\bar{A})} = \{x \in \overline{(\bar{A})}\} = \{x : x \notin \bar{A}\}$ by definition of complement

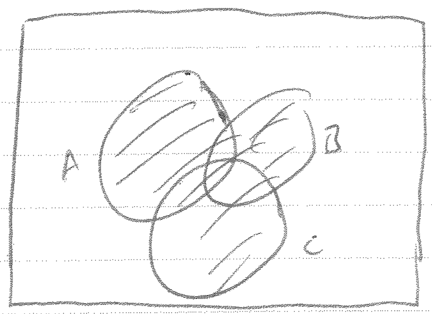
$= \{x : x \in A\}$ by definition of complement

$= A$ $\square$

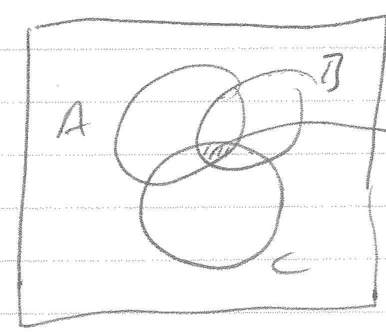
Also:

$A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$,

so can write $A \cup B \cup C$, $A \cap B \cap C$ without worrying about order.



$A \cup B \cup C$



$A \cap B \cap C$

Generalized Unions and Intersections

Defn ① The UNION of a collection of sets ~~$A_1, \dots$~~ is the set that contains those elements that are members of at least one set in the collection.

② The ~~Intersection~~ INTERSECTION of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

Write $A_1 \cup \dots \cup A_n = \bigcup_{i=1}^{n} A_i$, $A_1 \cap \dots \cap A_n = \bigcap_{i=1}^{n} A_i$

Eg: ⓐ $A_i = \{2i-1, 2i\}$ for $i = 1, 2, 3, \dots,$ ~~10~~

Then $\bigcup_{i=1}^{10} A_i = \{1, 2, \dots, 19, 20\}$

$\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, the set of positive integers.

ⓑ $A_i = \left\{\frac{1}{i}, \frac{2}{i}, \dots, \frac{i-1}{i}\right\}$, $i = 2, 3, 4, \dots$

so $A_2 = \{\frac{1}{2}\}$, $A_3 = \{\frac{1}{3}, \frac{2}{3}\}$, $A_4 = \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, ...

$\bigcup_{i=1}^{\infty} A_i =$ all rational numbers between 0 and 1.

ⓒ $A_i = \{i^0, i^1, i^2, \dots, i^i\}$ $i = 1, 2, 3, \dots$

so $A_1 = \{1, 1\} = \{1\}$ $A_2 = \{1, 2, 4\}$; $A_3 = \{1, 3, 9\}$, ...

$\bigcap_{i=1}^{\infty} A_i = \{1\}$.

Computer representation of sets

Don't want to store sets of objects any old how, because hard to do operations on the sets.

So take the universal set, and use a binary relationship to indicate if an element is in a set.

Eg: $U = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 3, 4\}$

Store A as 101100

U itself is stored as 111111

ϕ is stored as 000000

011011 corresponds to $B = \{2, 3, 5, 6\}$.

Easy to get $\overline{A}$: swap all the zeroes and ones, so 101100 becomes 010011, and $\overline{A} = \{2, 5, 6\}$.

Easy to get $A \cap B$: A is 101100
B is 011011

Compute: $A \cap B$ is 001000 so $A \cap B = \{3\}$
$A \cup B$ is 111011 so $A \cup B = \{1, 2, 3, 5, 6\}$

all done using bitwise operations

LOGIC AND PROOFS (Chapter 1)

PROPOSITIONAL LOGIC (§1.1)

A PROPOSITION is a declarative sentence.

Eg: ~~It will snow today~~
It snowed yesterday. ? TRUE ? FALSE

$1+4=5$ TRUE
$1+4=6$ FALSE

Denote a proposition by a letter, eg p.

NEGATION of a proposition denoted by $\neg p$

Eg: p is : $1+4=5$

$\neg p$ is : $1+4\neq5$.

CONJUNCTION of p and q: $p \wedge q$

Eg: p is "it snowed yesterday"

q is "it was sunny yesterday"

$p \wedge q$ is "it snowed and was sunny yesterday".

DISJUNCTION of p and q: $p \vee q$

Eg: $p \vee q$ is "it either snowed or it was sunny yesterday"

(Not exclusive-or, so if both p, q are true then $p \vee q$ is true)

EXCLUSIVE OR of p and q is denoted $p \oplus q$.

CONDITIONAL STATEMENT $p \rightarrow q$ is "if p then q"

Eg: p is "The quarterback plays well"

q is "The team wins"

$p \rightarrow q$ is "if the quarterback plays well then the team wins".

FALSE if ~~team~~ QB plays well and team loses
TRUE otherwise.

TRUTH TABLE

p	q	$p \vee q$	$p \wedge q$	$p \oplus q$	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T	F	T	T
T	F	T	F	T	F	T
F	T	T	F	T	T	F
F	F	F	F	F	T	T

Consider <u>all</u> cases of ~~the~~ combinations of the propositions p and q.

Propositional Equivalence (§1.2)

Defn ① A compound proposition that is always true is a TAUTOLOGY

② A compound proposition that is always false is a CONTRADICTION

Eg: ①: $p \vee \neg p$ is a tautology

②: $p \wedge \neg p$ is a contradiction.

De Morgan's Laws: ① $\neg(p \vee q)$ is logically equivalent to $\neg p \wedge \neg q$.

② $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$

Can be shown using a truth table:

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F	T	F	F
T	F	F	T	T	F	F	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	T	T	F	T	T	F	T	T

Various other equivalences that you should know are given in Table 6 on page 24.

Eg: $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Using De Morgan's Laws

Eg: p is "The Knicks are better than the Nets"

q is "The Yankees are better than the Mets"

$p \wedge q$ is "The Knicks are better than the Nets and the Yankees are better than the Mets."

The NEGATION of this is:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \text{ by De Morgan's Laws}$$

So the negation is:

"The ~~Yankees~~ Knicks are not better than the Nets

or

the Yankees are not better than the Mets."

So to show $p \wedge q$ is false only need to show ~~eg~~ either $\neg p$ or $\neg q$.

Predicates & Quantifiers (§1.3)

Look at statements involving variables

Eg: $\underbrace{P(x)}_{\text{propositional function}}$ is "x $\underbrace{\text{is prime}}_{\text{predicate}}$"

variable

$P(7)$ is TRUE

$P(9)$ is FALSE.

Defn The UNIVERSAL QUANTIFICATION of $P(x)$ is the statement:

"$P(x)$ for all values of x in the domain"

Denoted $\forall x \; P(x)$

Universal quantifier

Eg: $P(x)$ is "$2^x < x!$" for integers x.

If the domain is $x \geq 4$ then $\forall x \, P(x)$ is TRUE,

since indeed $2^x < x!$ for all $x \geq 4$.

If the domain is positive integers $x \geq 1$ then $\forall x \, P(x)$ is FALSE,

and $x = 3$ is a COUNTEREXAMPLE, since $2^3 = 8 > 6 = 3!$

DEFN The EXISTENTIAL QUANTIFICATION of $P(x)$ is the proposition
"There exists an element x in the domain such that $P(x)$."

Notation: $\exists x\, P(x)$

$\exists$ — existential quantifier

Eg: $P(x)$ is "x is prime"

If domain is positive integers then $\exists x P(x)$ is TRUE, eg, $x = 7$

If domain is even integers ≥ 4 then $\exists x P(x)$ is FALSE.

Negating an existential quantifier

$\neg \exists x\, P(x)$

There does not exist an x so that $P(x)$ is true.

Equivalently. For all x, $P(x)$ is false. $\forall x \neg P(x)$.

So $\neg \exists x P(x) \equiv \forall x \neg P(x)$

Similarly: $\neg \forall x\, P(x) \equiv \exists x \neg P(x)$

Generalize De Morgan's Laws.

Nested Quantifiers (§1.4)

Two quantifiers are NESTED if one is within the scope of the other

Eg: let ~~x~~ the domains of x and y be the reals.

① $\forall x \exists y \, (x+y=5)$

TRUE: for any given x, choose $y = 5-x$.

② $\forall x \exists y \, (xy=1)$

FALSE. Take $x=0$.

Negating a nested quantifier:

~~Use De Morgan's Laws:~~

$\neg(p \wedge q) \equiv \neg p \vee \neg q$, $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (On previous page)

~~Extend to multiple propositions:~~

$\cancel{\neg(P(x))}$ $\neg(\forall x \, P(x)) \equiv \exists x \, \neg P(x)$

$\neg \forall x \exists y \, (x+y=5)$

$\equiv \exists x \, \neg \exists y \, (x+y=5)$

$\equiv \exists x \, \forall y \, \neg(x+y=5)$ There exists an x so that for all y we have $x+y \neq 5$

FALSE: take $y = 5-x$

Similarly:

$\neg \forall x \exists y \, (xy=1) \equiv \exists x \, \neg \exists y \, (xy=1)$

$\equiv \exists x \, \forall y \, \neg(xy=1)$ TRUE: $x=0$ and then $\forall y$ we have $xy=0$, so $xy \neq 1$.

Rules of Inference (§1.5)

Valid Arguments

p: "You get 100% on the final." q: "You get an A."

If we know $p \to q$ is TRUE
and p is TRUE
then q is TRUE

MODUS PONENS (law of detachment)

Other valid arguments:

$\neg q$ and $p \to q$: $\therefore \neg p$ (If it's Tuesday then I have Discrete Structures. I don't have Discrete Structures today. $\therefore$ it's not Tuesday.)

$p \to q$ and $q \to r$: $\therefore p \to r$ (If it's hot I'll be thirsty. If I'm thirsty I'll drink water. $\therefore$ If it's hot I'll drink water.)

$p \vee q$ and $\neg p$: $\therefore q$ (I'm either cold or thirsty. I'm not cold. $\therefore$ I must be thirsty)

p : $\therefore p \vee q$ (I'm cold $\therefore$ I'm either cold or thirsty)

$p \wedge q$: $\therefore p$ (I'm cold and thirsty. $\therefore$ I'm cold)

p and q : $\therefore p \wedge q$

$p \vee q$ and $\neg p \vee r$: $\therefore q \vee r$ (RESOLUTION)

RESOLVENT.

Eg: p is "$x \le 3$" q is "$2^x < x!$" r is "$2^x \le 8$"

$p \vee q$ is "Either $x \le 3$ or $2^x < x!$" $\neg p \vee r$ is "Either $x > 3$ or $2^x \le 8$"

$q \vee r$ is "Either $2^x < x!$ or $2^x \le 8$."

Fallacies

Affirming the conclusion:

$$[(p \to q) \wedge q] \to p \quad \text{NOT VALID.}$$

Eg: p is "The QB plays well." q is "The team wins."

Can have the team win and the QB play poorly.

Then $p \to q$ and q are both TRUE

but p is FALSE.

Denying the hypothesis:

$$[(p \to q) \wedge \neg p] \to \neg q \quad \text{NOT VALID}$$

Eg: p, q as above. As before, if the team wins and the QB plays poorly

then $p \to q$ is TRUE and $\neg p$ is TRUE but $\neg q$ is FALSE.

Eg: For every x, we know that if $\underbrace{x \geq 4}_{P(x)}$ then $\underbrace{2^x < x!}_{Q(x)}$

$$\forall x \, (P(x) \to Q(x))$$

We know $x = 5$.

Therefore $2^x < x!$ Universal Modus ponens

Introduction to Proofs (§1.6)

Direct Proofs

Assume the hypothesis is true.

Construct subsequent steps using rules of inference.

Eventually get to conclusion.

Eg: Theorem If n is an ~~Every~~ odd integer then it is the difference of two squares

Proof Hypothesis: we have an odd integer.

Want to show: it is the difference of the square of two numbers.

Since n is odd, we can write $n = 2k+1$ for some integer k.

Now, $(k+1)^2 = k^2+2k+1 = (k^2)+(2k+1)$.

So $n = (k+1)^2 - k^2$ $\square$

Eg: If n is odd then n^2+4 is odd.

Proof by Contraposition:

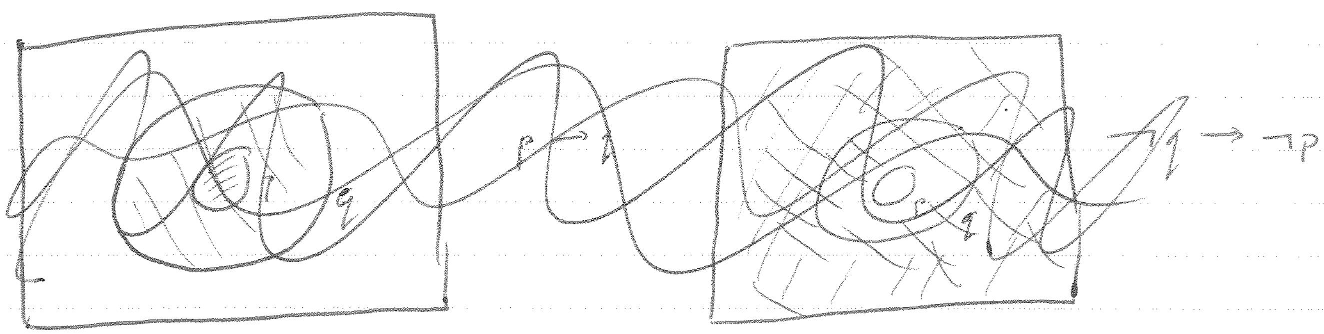
Assumption p. Want to conclude q.

Now $p \to q \equiv \neg q \to \neg p$

So: assume q is FALSE. Show p must be false.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	~~T~~ F	F
F	T	T	F	~~F~~ T	T
F	F	T	T	T	T

Columns are identical.



Eg: If the real numbers x and y satisfy $x + y \geq 2$ (p)

then either $x \geq 1$ or $y \geq 1$. (q)

<u>Proof</u> Use proof by contraposition.

$\neg q$ is $x < 1$ and $y < 1$. Then $x + y < 2$. Contradicts hypothesis.

So if $x + y \geq 2$ then either $x \geq 1$ or $y \geq 1$ (or both). ∎

See next page for another example.

Proof by contradiction

Want to show p is true.

Suppose we know a q such that $\neg p \to q$, and we know that q is false.

Then $\neg p$ is false, so p is true.

Eg: If three socks are picked from a drawer containing blue and red socks then a pair of socks must be chosen. (PIGEONHOLE PRINCIPLE)

Proof p is "at least two of the three chosen socks have the same color."

But for all the socks to be different, we must have no more than two socks.

Contradiction. $\blacksquare$

Eg: Show that if a is less than b then the average of a and b is greater than a (Proof by contraposition)

Proof p: a is less than b

q: the average of a and b is greater than a.

Assume p holds and $\neg q$ holds. Get a contradiction.

Let $m = \frac{a+b}{2}$. We assume $\neg q$, so $m \leq a$

Thus, $a + b = 2m \leq 2a$, so $b \leq a$.

Contradiction to p.

Eg: <u>Thm</u> If n is an integer and n^2+5 is odd then n is even.

<u>Proof</u> p: ~~n is an integer with~~ n^2+5 odd
q: n even.

Assume p and $\neg q$ both hold. Get a contradiction.

So assume n^2+5 is odd and n is odd.

Since n is odd, we have $n=2k+1$ for some integer k.

Then $n^2=(2k+1)^2=4k^2+4k+1$

so $n^2+5 = 4k^2+4k+6 = 2(2k^2+2k+3)$, even

$2l+1$ by assumption p.
odd

Contradiction.

Mistakes in proofs

Eg. Prove $1 = 2$:

"Proof": Let a and b be two equal integers.

So: $a = b$

Then: $a^2 = ab$

and $a^2 - b^2 = ab - b^2$

so $(a-b)(a+b) = (a-b)b$.

Divide by $a-b$:

$a + b = b$

So $2b = b$ since $a = b$

and thus: $2 = 1$ divide by b.

Problem: can't divide by $a-b$ since $a-b=0$.

Proof Methods & Strategy (§1.7)

Proof by Cases

Eg: Triangle inequality

For any two real numbers, we have $|x| + |y| \geq |x+y|$

Proof: Look at cases depending on the sign of x, y:

$x \geq 0, y \geq 0$: $|x| + |y| = x+y$. $|x+y| = x+y$ ✓

$x \leq 0, y \leq 0$: $|x| + |y| = -x-y$, $|x+y| = -x-y$ ✓

$x \geq 0, y < 0, x+y \geq 0$: $|x|+|y| = x-y$, $|x+y| = x+y < x-y$ since $y<0$ ✓

$x \geq 0, y<0, x+y<0$: $|x|+|y| = x-y$, $|x+y| = -y-x \leq x-y$ since $x \geq 0$ ✓

$x<0, y \geq 0, x+y \geq 0$: $|x|+|y| = -x+y$, $|x+y| = x+y \leq -x+y$ since $x<0$ ✓

$x<0, y \geq 0, x+y<0$: $|x|+|y| = -x+y$, $|x+y| = -x-y \leq -x+y$ since $y \geq 0$ ✓

∎

Existence Proof

Constructive: Eg: Prove there is a positive integer ~~st~~ greater than 1 that is a perfect square and a perfect cube.

Proof: Find such a number.

Eg: $64 = 4^3 = 8^2$

$729 = 9^3 = 27^2$

$\blacksquare$

Nonconstructive:

Eg: Given a number n, there exists a prime number larger than n.

Proof Assume not. Let $p_1, \ldots, p_k$ be all the primes less than or equal to n.

Let $p = p_1 p_2 p_3 \cdots p_k$.

If $p \leq n$, let $\bar{p} = p p_k^r$ with r the smallest integer so that $\bar{p} > n$.

Else, let $\bar{p} = p$.

Now, take $q = \bar{p} + 1$.

None of the prime numbers smaller than n is a factor of q.

So either q is prime or q has a ~~fa~~ prime factor that is greater than n.

$\blacksquare$

Uniqueness

Eg: A square matrix M is invertible if there exists another matrix A satisfying $MA = I$ $\left(\text{and } AM = I\right)$.

Thm An invertible matrix M has exactly one inverse.

Proof By definition, M has at least one inverse A.

Assume C is also an inverse of M.

So $MA = AM = I$ by definition

and $MC = CM = I$ by assumption

Then
$$CMA = (CM)A = A$$
$$CMA = C(MA) = C$$

So $C = A$. $\blacksquare$

Counterexamples

Eg: Let A, B, C be three sets.

① Assume $A \cap B = A \cap C$. Is $B = C$?

Counterexample: B, C can contain any combo of elements in $\overline{A}$.

② Assume $A \cup B = A \cup C$. Is $B = C$?

Counterexample: B, C can contain any combo of elements in A

③ Assume $A \cap B = A \cap C$ and $A \cup B = A \cup C$. Is $B = C$?

Yes: Proof:

Let $x \in B$. Want to show $x \in C$.

Look at cases:

① $x \notin A$.

Now, $x \in A \cup B$ since $x \in B$

so $x \in A \cup C$ since $A \cup B = A \cup C$

but $x \notin A$.

So $x \in C$ ✓

② $x \in A$

So $x \in A \cap B$ since $x \in B$

So $x \in A \cap C$ since $A \cap B = A \cap C$

So $x \in C$ ✓

$\boxed{///}$

Proof strategies:

Analyze what the hypothesis & conclusion mean.

Try ~~forward~~ direct proof, contrapositive, contradiction, proof by cases.

Look for counterexamples if stuck.

Use "backward reasoning": if we want this conclusion, how could we get here?

① Eg: (Q20) Show that if x is a nonzero real number then $x^2 + \frac{1}{x^2} \geq 2$.

Now, $x^2 + \frac{1}{x^2} \geq 2$ is equivalent to:

$$x^2 - 2 + \frac{1}{x^2} \geq 0 \quad \text{for nonzero } x$$

~~or~~ $\cancel{(x^2-1)^2 \geq 0}$

or $\left(x - \frac{1}{x}\right)^2 \geq 0$, which is always true for nonzero x.

Turn this into forward reasoning:

If x is nonzero then $\left(x - \frac{1}{x}\right)^2 \geq 0$ so $x^2 + \frac{1}{x^2} - 2 \geq 0$

so $x^2 + \frac{1}{x^2} \geq 2$ $\square$

② Eg: Let x, y be reals. Show $x^2 + 25y^2 \geq 10xy$

Proof: Work backwards. How to get $x^2 + 5y^2 \geq 10xy$?

Equivalent to $x^2 - 10xy + 25y^2 \geq 0$

Equivalent to $(x - 5y)^2 \geq 0$ $\square$

Can turn into forward reasoning.