INTRODUCTION TO DIFFERENTIAL EQUATIONS, TEST 1
Sections 9-12, Fall 2006

Instructions. You are allowed to use one 8 1/2 × 11 inch sheet of paper of notes. No calculators, PDAs, computers, books, or cellular phones are allowed. Do not collaborate in any way. In order to receive credit, your answers must be clear, legible, and coherent.

1. (2 points each) Fill in the blank with the letters corresponding to the best description. Use the abbreviations SHM = simple harmonic motion; OD = overdamped; CD = critically damped; UD = underdamped; E = exponentially growing.

i) $CO \ddot{u} + 4\dot{u} + 4u = 0$ Try $u = e^{rt}$, $r^2 + 4r + 4 = 0$. This is $(r + 2)^2 = 0$. $r = -2$.

ii) $SHM \ddot{u} + 4u = 0$ $r^2 + 4 = 0$ $r = \pm 2i$.

iii) $UD \ddot{u} + 2\dot{u} + 4u = 0$ $r^2 + 2r + 4 = 0$. $r = \frac{-2 \pm \sqrt{4 - 4 \cdot 4}}{2} = -1 \pm \sqrt{3}i$.

iv) $OD \ddot{u} + 4\dot{u} + 2u = 0$ $r^2 + 4r + 2 = 0$. $r = \frac{-4 \pm \sqrt{16 - 4 \cdot 2}}{2} = -2 \pm 2\sqrt{2}$.

v) $E \ddot{u} + 4u = 0$ $r^2 - 4 = 0$ $r = \pm 2$.

vi) $E \ddot{u} - 4\dot{u} + 4u = 0$ $r^2 - 4r + 4 = 0$. $(r - 2)^2 = 0$. $r = 2$.

vii) $CO$

viii) $SHM$

ix) $UD$

x) $E$

2. (5 points) The solution of a certain spring-mass problem is $u(t) = \cos t - \sin t$.

For this solution, the angular frequency is $\boxed{1}$, the period is $\boxed{2\pi}$, and the amplitude is $\boxed{\sqrt{2}}$.

$$u = A \cos(t - \delta) = A(\cos t \cos \delta + \sin t \sin \delta)$$

$1 = A \cos \delta = 1$ \hspace{1cm} $A \sin \delta = -1$

$$A^2 (\cos^2 \delta + \sin^2 \delta) = 1 + 1 = 2$$

$$A = \sqrt{2}$$
3. For the equation \( dy/dt = y(y + 1)(y - 3) \):
   a) (3 points) sketch a graph of \( dy/dt \) as a function of \( y \).

   ![Graph of dy/dt as a function of y]

   b) (3 points) Identify the equilibrium solutions and determine the stability of each.
   
   \[
   \begin{array}{c|c}
   y = -1 & \text{unstable} \\
   y = 0 & \text{stable} \\
   y = 3 & \text{unstable}
   \end{array}
   \]

   c) (4 points) sketch a graph of \( y \) as a function of \( t \) for the initial condition \( y(0) = 2 \).

   ![Graph of y as a function of t]

4. (10 points) Solve the initial-value problem \( yy' + \sin t = 0 \), \( y(0) = -2 \).

   \[
   \frac{dy}{dt} = -\frac{\sin t}{y}
   \]

   \[
   \int y \, dy = -\int \sin t \, dt + C
   \]

   \[
   \frac{1}{2} y^2 = \cos t + C
   \]

   \[
   y = \pm \sqrt{2(\cos t + C)}
   \]

   \[
   y(t) = -\sqrt{2(\cos t + 1)}
   \]
5. (15 points) Solve the initial-value problem \( y'' + 2y' + 4y = 0; \ y(0) = 0; \ y'(0) = 2\sqrt{3}: \)
(Hint: This is the complex-root case)

Try \( y = e^{\tau t} \quad r^2 + 2r + 4 = 0 \quad r = -2 \pm \frac{\sqrt{4 - 4 \cdot 4}}{2} = -1 \pm \sqrt{1-4} = -1 \pm \sqrt{3} i, \)

\[ y(t) = e^{-t} \left( c_1 e^{\sqrt{3} i t} + c_2 e^{-\sqrt{3} i t} \right) \]

The general solution is \( y(t) = c_1 e^{-t} \cos \sqrt{3} t + c_2 e^{-t} \sin \sqrt{3} t \)

\[ 0 = y(0) = e_1 (1)(1) + c_2 (1)(0) = c_1 \]

\[ 2\sqrt{3} = y'(0) = c_2 (1)(1) + (1) \sqrt{3} (1) = \sqrt{3} c_2 \]

\[ \Rightarrow c_2 = 2 \]

\[ y(t) = 2 e^{-t} \sin \sqrt{3} t \]

6. (10 points) For the initial-value problem \( y' + 2y \cos t + \sin t = 0; \ y(0) = 1: \)

a) The slope of the solution \( y(t) \) at \( t = 0 \) is \( -2 \)

\[ e^{0} \quad y = 1, \quad y' = -2(1)(1) + (0) = -2 \]

b) An approximation to \( y(0.1) \) (from taking one step of the Euler method) is \( 0.8 \)

\[ y(0.1) = y(0) + y'(0)(0.1) = 1 - 2(0.1) = 1 - 0.2 = 0.8 \]
7. (30 points) In your chemistry lab, students start with a 4-liter beaker of pure water. When a valve is opened, a 50% solution of compound \( X \) flows in at the rate of four liters per minute, and the mixed solution flows out at the same rate. Before the valve is opened, you go into the hall for a few minutes to take a call on your cell phone. When you return, your lab partner has already opened the valve and the experiment is running. You sample the fluid flowing out and discover that the concentration of compound \( X \) is 25% in the effluent. How long ago did your lab partner start the experiment? (You may leave your answer in terms of fractions, exponentials, and logarithms.)

a) The differential equation to solve is \( \frac{dX}{dt} = \frac{2 - X}{4} \)

(If you have trouble with this, just use \( \frac{dX}{dt} = a - bX \) for the questions below.)

\[
\frac{dX}{dt} = (\text{rate of concentration in}) - (\text{rate of concentration out})
\]

\[
= (0.5)(4 \text{ l/min}) - \frac{X}{4} \text{ l/min} = 2 - X
\]

b) The initial condition is \( X(0) = 0 \)

\[ \frac{dX}{dt} + X = 2 \]

c) The integrating factor is \( e^t \)

d) The general solution is \( X(t) = 2 + C e^{-t} \)

\[
\frac{d}{dt} (e^t X) = 2 e^t
\]

\[
e^t X = 2 \int e^t dt + C
\]

\[
X = 2 + C e^{-t}
\]

e) The solution for the initial condition in b) is \( X(t) = \frac{2 - 2 e^{-t}}{2} \)

\[ 0 = X(0) = 2 + C \quad \Rightarrow \quad C = -2 \]

f) The experiment has been running for \( t = \frac{\ln 2}{2} \) minutes.

\[
\therefore 2 e^{-t} = \frac{1}{4}
\]

\[
e^{-t} = \frac{1}{2} \quad \Rightarrow \quad e^t = 2 \quad \Rightarrow \quad t = \ln 2
\]